### 9.4.1 Application to the rotating earth

The above theory of the symmetrical top can be applied to the rotation of the earth. The earth is known to be slightly flattened near the poles because of which it can regarded as an oblate spheroid. This gives $I_1 = I_2 \cong I_3$, and $I_3 > I_1$. Therefore the angular frequency

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

is much smaller than $\omega_3$, such that $\Omega \cong \omega_3/300$. Since the period of Earth's rotation is $1/\omega = 1$ day and $\omega_3 \cong \omega$, we get $(1/\Omega = 300\text{days})$

$$T_p = \frac{2\pi}{\Omega} = \frac{2\pi I_1}{\omega_3(I_3 - I_1)} = \frac{1\text{day}}{0.00327} = 305 \text{ days}$$

The measured value is $\cong$ 440 days. The difference can be explained by noting that the earth is not a perfect sphere, neither is it strictly a rigid body.

## 9.5 General Motion of a Rigid Body

In the general motion of the body, (i.e. no point of the body is fixed in space), let $\mathbf{F}_c^{ext}$ be the total external force on the rigid body and $\mathbf{G}_c^{ext}$ the total external torque about its mass centre (i.e. centroid). Then the equations of motion are

$$M\mathbf{a}_c = \mathbf{F}_c^{ext} \tag{9.5.1}$$

and

$$\dot{\mathbf{L}}_c = \mathbf{G}_c \tag{9.5.2}$$

where $\mathbf{a}_c$ is the acceleration of the c.m. and $\mathbf{L}_c$ is the total angular momentum about it. Now we resolve the vectors $\mathbf{a}_c$, $\mathbf{F}$, $\mathbf{G}_c$ and $\mathbf{L}$ along the unit vector $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ taken along the principal axes at the mass centre. The triad of vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ may be referred to as a principal triad. It will be assumed to be permanently a principal triad. Let $\vec{\Omega}$ be its angular velocity. If the triad is fixed in the body then $\vec{\Omega} = \vec{\omega}$, the angular velocity of the body.

Now using the relation

$$\left(\frac{d\mathbf{F}}{dt}\right) = \left(\frac{d\mathbf{F}}{d}\right) + \vec{\omega}\times\mathbf{F}$$

which relates the rates of change of a vector in a fixed (i.e. inertial) frame and a rotating frame, we have (on dropping the suffix r) $(\mathbf{v}_f = \mathbf{v}_r = \mathbf{v})$

$$\mathbf{a}_f \equiv \left(\frac{d\mathbf{v}}{dt}\right)_f = \frac{d\mathbf{v}}{dt} + \vec{\Omega}\times\mathbf{v},$$

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$(\mathbf{v}_f = \mathbf{v}_r = \mathbf{v})$

where $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ is the velocity of the mass centre (in the r... coordinate system). Substituting for $\mathbf{a}_f=\mathbf{a}_c$ from (9.5.3) into (9... obtain

$$M\left(\frac{d\mathbf{v}}{dt}+\vec{\Omega}\times\mathbf{v}\right)=\mathbf{F}$$

which is equivalent to

$$\left.\begin{aligned} M(\dot{v}_1+\Omega_2 v_3-\Omega_3 v_2) &= F_1\\ M(\dot{v}_2+\Omega_3 v_1-\Omega_1 v_3) &= F_2\\ M(\dot{v}_3+\Omega_1 v_2-\Omega_2 v_1) &= F_3 \end{aligned}\right\} \quad (9$$

From (9.5.2), on using

$$\left(\frac{d\mathbf{L}}{dt}\right)_f = d\mathbf{L}dt+\Omega\times\mathbf{L}$$

and the relation $\mathbf{L}=I_1\omega_1\mathbf{i}_1+I_2\omega_2\mathbf{j}+I_3\omega_3\mathbf{k}$, we obtain the equa...

$$\begin{aligned} I_1\dot{\omega}_1\mathbf{i} + I_2\dot{\omega}_2\mathbf{j}+I_3\dot{\omega}_3\mathbf{k} + (\Omega_2 L_3-\Omega_3 L_2)\mathbf{i}\\ + (\Omega_2 L_1-\Omega_1 L_3)\mathbf{j} + (\Omega_1 L_2-\Omega_2 L_1)\mathbf{k} = \mathbf{G} \end{aligned}$$

From this vector equation we obtain the following three scalar equa...

$$I_1\dot{\omega}_1 + \Omega_2 L_3 - \Omega_3 L_2 = G_1$$

$$I_2\dot{\omega}_2 + \Omega_3 L_1 - \Omega_1 L_3 = G_2$$

and

$$I_3\dot{\omega}_3 + \Omega_1 L_2 - \Omega_2 L_1 = G_3$$

where we have used the results

$$L_1=I_1,\quad L_2=I_2\omega_2,\quad L_3=I_3\omega_3$$

we have

$$\left.\begin{aligned} I_1\dot{\omega}_1+\omega_3\Omega_2 I_3-\omega_2\Omega_3 I_2 &= G_1\\ I_2\dot{\omega}_2+\omega_1\Omega_3 I_1-\omega_3\Omega_1 I_3 &= G_2\\ I_3\dot{\omega}_3+\omega_2\Omega_1 I_2-\omega_1\Omega_2 I_1 &= G_3 \end{aligned}\right\} \quad (9.5.5$$

which are the same as for a rigid body with a fixed point. In these equa... $I_1$, $I_2$, $I_3$ denote principal moments of inertia at the centroid of the ... The sets of equations (9.5.4) and (9.5.5) constitute six equations fo... components of velocity of the mass centre and the components of an... velocity of the body. For any of these six equations, we can substitut... law of conservation of energy, *viz.* $T+V=E$, provided the external fo... are conservative. The last equation is equivalent to

$$\frac{1}{2}M(v_1^2+v_2^2+v_3^2)+\frac{1}{2}(I_1\omega_1^2+I\ldots$$

## 9.6 Stability of Rigid Body Rotations

The problem of stability of rotations of a rigid body was first studied by Euler in 1749. We will assume that there is no external force on the rigid body and it is rotating about one of its principal axes. The motion of the rigid body will be deemed to be stable if under a small perturbation the body will return to its former state of motion or will perform small oscillations about the fixed point (or axis).

Let $I_1$, $I_2$, $I_3$ denote the principal moments of the body and without loss of generality we suppose that $I_1 < I_2 < I_3$. We choose the body coordinate system along the principal axes and take the body axis $OX_1$ corresponding to the principal moment $I_1$, as the axis of rotation. Then the angular velocity $\vec{\omega}$ of the body can be represented as

$$\vec{\omega} = \omega_1 \mathbf{i} \tag{9.6.1}$$

When a small perturbation is applied, the axis of rotation is slightly displaced and the angular velocity then takes the form

$$\vec{\omega} = I_1 \mathbf{i} + \lambda \mathbf{j} + \mu \mathbf{k} \tag{9.6.2}$$

where $\lambda$, $\mu$ are very very small parameters. The Euler dynamical equations are

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0 \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0 \end{aligned}$$

For the problem under discussion, from (9.6.2), $\omega_2 = \lambda$, $\omega_3 = \mu$ and therefore the Euler equations become

$$\left.\begin{aligned} I_1 \dot{\omega}_1 + (I_2 - I_3)\lambda\mu &= 0 \\ I_2 \dot{\lambda} + (I_3 - I_1)\mu\omega_1 &= 0 \\ I_3 \dot{\mu} + (I_1 - I_2)\omega\lambda &= 0 \end{aligned}\right\} \tag{9.6.3}$$

Since the produce $\lambda\mu$ is negligibly small, the first of equations (9.6.3) reduces to $\dot{\omega}_1 = 0$ or $\omega_1 =$ constant. From the second and the third equations of (9.6.3) we obtain

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu \tag{9.6.4}$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\lambda \tag{9.6.5}$$

where each term in the parentheses is constant. Differentiating equations w.r.t. $t$ and eliminating $\dot{\lambda}$ or $\dot{\mu}$ with the help of the obtain the second order differential equations

$$\ddot{\lambda}+\frac{(I_1-I_3)(I_1-I_2)}{I_2 I_3}\omega_1^2\lambda = 0$$

and

$$\ddot{\mu}+\frac{(I_1-I_2)(I_1-I_3)}{I_2 I_3}\omega_1^2\mu = 0$$

Mathematically equation (9.6.7) is exactly the same as (9.6.6) w placed by $\mu$.

The solution of (9.6.7) is given by

$$\lambda(t) = Ae^{i\Omega_1 t}+Be^{-i\Omega_1 t}$$

where

$$\Omega_1^2 = \frac{(I_1-I_2)(I_1-I_3)}{I_2 I_3}\omega_1^2$$

From our assumption that $I_1 < I_2 < I_3$, it follows that $\Omega_1$ is re solution (9.6.8) therefore represents oscillatory motion with a fr $\Omega_1$. The solution for $\mu(t)$ is the same as in (9.6.8) i.e.

$$\mu(t) = Ae^{i\Omega_1 t}+Be^{-i\Omega_1 t}$$

From (9.6.8) and (9.6.8′) we conclude that the small perturbative in (9.6.2) do not increase with time but oscillate around the equil values $\lambda = 0$, and $\mu = 0$. Hence the rotation about the $OX$- axis is stab

Similarly if we consider the rotations about the $OY$ and $OZ$-axes, then corresponding angular frequencies $\Omega_2$ and $\Omega_3$ can be obtained from as

$$\Omega_2^2 = \frac{(I_2-I_3)(I_2-I_1)}{I_3 I_1}\omega_2^2$$

and (9

$$\Omega_3^2 = \frac{(I_3-I_1)(I_3-I_2)}{I_1 I_2}\omega_3^2$$

(9.

Since $I_1 < I_2 < I_3$ we find that $\Omega_1$, $\Omega_3$ are real whereas $\Omega_2$ is pure imaginary.

It follows that if the rotation takes place around the $OX$ or $OZ$- axis, th perturbation produces oscillatory motion and the rotation is stable. If rotation takes place around the $OY$-axis, because of $\Omega_2$ being imagin the exponential factor $e^{-i\Omega_2 t}$ in the solution for $\lambda(t)$ and $\mu(t)$ beco the matrix

$$R_\phi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.7.1}$$

The angle $\phi$ is called *precession angle.* After applying this transformation, the new coordinate system is denoted by $OX'Y'Z'$, and the relation between the coordinates is given by

$$\mathbf{X}' = R_\phi \mathbf{X}_0 \tag{9.7.2}$$

where $\mathbf{X}_0$ denotes the column vector of coordinates i.e. $[x_0,\ y_0,\ z_0]^t$. The column vector $\mathbf{X}'$ has a similar definition.

2. The second rotation takes place in the $Y'Z'$-plane, in the counterclockwise direction about the $OX'$-axis through an angle $\theta$. The rotation matrix in this case is given by

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{9.7.3}$$

The angle $\theta$ is called *nutation* angle. The new coordinate system is now denoted by $OX''Y''Z''$, and the coordinates are related by

$$\mathbf{X}'' = R_\theta \mathbf{X}' \tag{9.7.4}$$

3. The third rotation takes place in the $OX''Y''$-plane in the counterclockwise direction through an angle $\psi$ about the $OZ''$-axis. This transformation brings us to the body coordinate system $OXYZ$. The rotation matrix in this case is given by

$$R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.7.5}$$

and the corresponding coordinate vectors are related by

$$\mathbf{X} = R_\psi \mathbf{X}'' \tag{9.7.6}$$

The angle $\psi$ is called the *body angle.* The transformation from the fixed coordinate system $OX_0Y_0Z_0$ to the body coordinate system $OXYZ$ (see figure 9.5) is given by the rotation matrix $R = R_\psi R_\theta R_\phi$, which when written in full becomes

$$R = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ \dots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ \dots \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\omega}_\phi(b) = R_\psi R_\theta \vec{\omega}_\phi = R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix}$$

The rotation about $OZ''$ axis through angle $\psi$ is the same as rotation a
$OZ$ through the same angle. Hence $\vec{\omega} = \dot{\psi}\mathbf{k}'' = \dot{\psi}\mathbf{k}$. Now

$$\vec{\omega} = \vec{\omega}_\theta(b) + \vec{\omega}_\phi(b) + \vec{\omega}_\psi(b)$$

or

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

which gives

$$\omega_1 = \dot{\phi}\sin\theta\sin\phi + \dot{\phi}\cos\psi \qquad (9.7.7a.)$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad (9.7.7b.)$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \qquad (9.7.7c.)$$

These equations are called *Euler's geometrical equations*. They descr
rigid body motion relative to the body coordinate system.

### Note on Notation

The notation used by the British authors is different from the one us
here. We have adopted the notation used by American authors, (Goldste
in particular). Our $\phi$ and $\psi$ are equal to $\phi + \pi/2$ and $\pi/2 - \psi$ in the British
notation. Some other authors use $\phi$ instead of $\psi$ and vice versa.

## 9.8 Tops and Gyroscopes

Motions of toy tops are quite frequently seen in everyday life. It's alway
fascinating to observe the spinning motion of a top along with its pre
cession, its rise, its sleep and finally its death. The theory of spinnin

top has relevance in many areas of practical life in applied mechanics (gyroscopic instruments), atomic, molecular and nuclear physics (a whirling molecule/atom or nucleus), and in Astronomy ( a spinning planet etc. )

A top is called *sleeping* if it is spinning about its axis of symmetry, which is vertical.

Mathematically gyroscope or top is a rigid body symmetrical about an axis and rotating about that axis. (When the gyroscope rotates about a fixed axis, the angular momentum vector of the gyroscope, about a point on the axis of rotation, is directed along the axis of rotation.) However in applied mechanics gyroscope is a specific device.

Rapidly rotating and heavy bodies are very stable. This fact is the basis of the*gyroscope*. Essentially this consists of a spinning body suspended in such a way that its axis is free to rotate relative to its support. The bearing are designed to be nearly frictionless so that the effect of torque due to the friction is nearly zero. When this is the case, then no matter how use turn the support, the axis of the gyroscope will remain pointing very closely to the same direction in space. More detailed description of the gyroscope is given below.

It consists of a heavy rotating fly wheel, which is mounted in such a way that its axis can freely change direction. This can be achieved by supporting it on a universal joint, or more usually, in what is called*gimbal mounting*. This consists of an outer and an inner ring. The outer ring turns

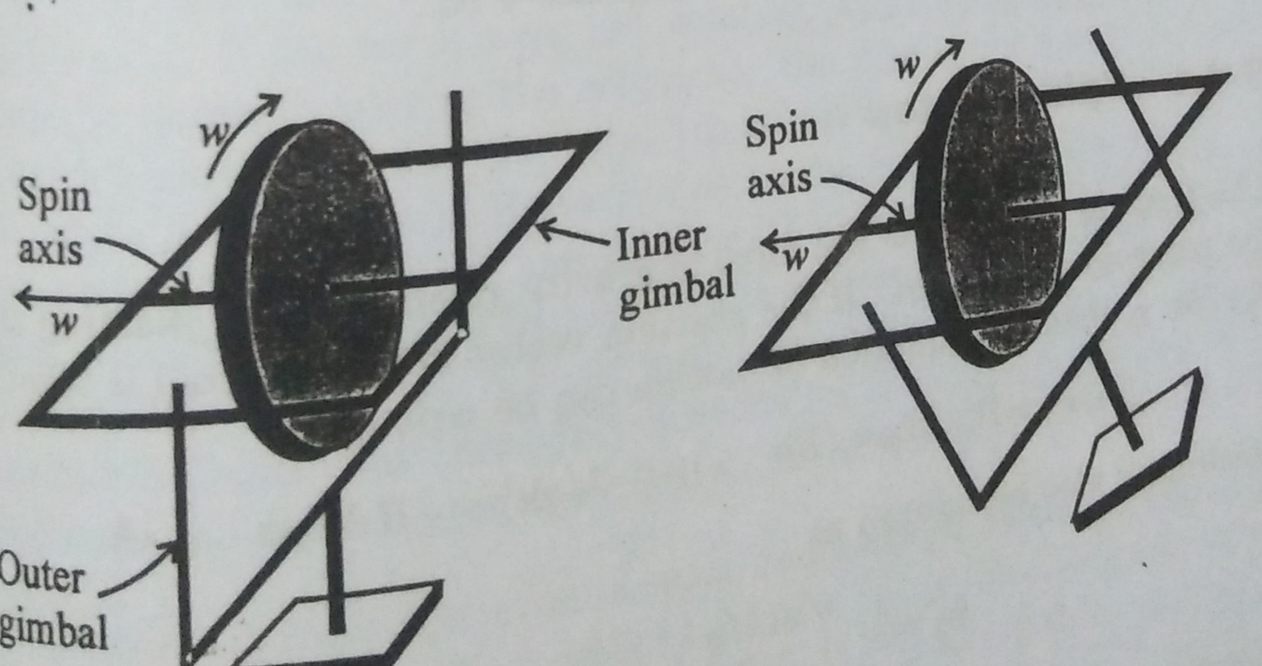


Figure 9.6:

freely about a vertical axis fixed to an external support, while the inner ring turns freely about a horizontal axis fixed to the outer ring. The flywheel rotates about an axis fixed to the inner ring, which is at right angles to both the other axes. As a result of this arrangement, any torque on the external support does not transfer itself to the flywheel, which continues to point in the same direction in space. Further, if there is a little friction in the bearing, which transfer part of the torque, the gyroscopic effect mentioned above takes care of this decrease in the torque. For this reason, the arrangement is used in inertial guiding systems in ships and aeroplanes.

In a *gyroscope* the inner and outer rings are fixed to each other, and the external casing is arranged to move freely in a horizontal plane.

The stability induced by the spin about the symmetry axis is called the *gyroscopic effect*; since it is this principle on which the working of a gyroscope is based. This principle is used, among other things, in the construction of the barrel of a rifle. The barrel of a rifle has a helical groove cut into it. This makes the bullet move along its axis, which ensures that it continuous to point in the direction of its motion after leaving the barrel.

The importance of the gyroscope as a directional stabilizer arises from the fact that the angular momentum vector $\mathbf{L}$ remains constant when the torque is applied. The changes in the direction of a well-made gyroscope are small because the applied torques are small and $\mathbf{L}$ is very large, so that $d\mathbf{L}/dt$ gives no appreciable change in direction. Moreover, a gyroscope only changes direction while a torque is applied. If it shifts slightly due to occasional small frictional torques in its mountings, it stops shifting when the torque stops. A large non-rotating mass, if mounted like a gyroscope, would acquire only small angular velocities due to frictional torques, but once set in motion by a small torque, it would continue to rotate, and the change in position might become large as $t \to \infty$.

### 9.8.1 Spinning top

The force in this case is the force of gravity given by $\mathbf{F} = Mg\mathbf{k}$ acting at the centroid of the top $C$. If the position vector of the centroid is taken as $\mathbf{R} = R\mathbf{e}_3$, then the equation of motion can be written as

$$I_3 \omega \dot{\mathbf{e}}_3 = \mathbf{R} \times M\mathbf{g} = (R\mathbf{e}_3) \times (-Mg\mathbf{k}) = -RMg\mathbf{e}_3 \times \mathbf{k}$$

which may also be written as

$$\dot{\mathbf{e}}_3 = -\left(\frac{RMg}{I_3\omega}\right)\mathbf{e}_3 \times \mathbf{k} = \vec{\Omega} \times \mathbf{e}_3 \qquad (9.8.1)$$

where

$$\vec{\Omega} = \frac{RMg}{I_3\omega}\mathbf{k} \qquad (9.8.2)$$