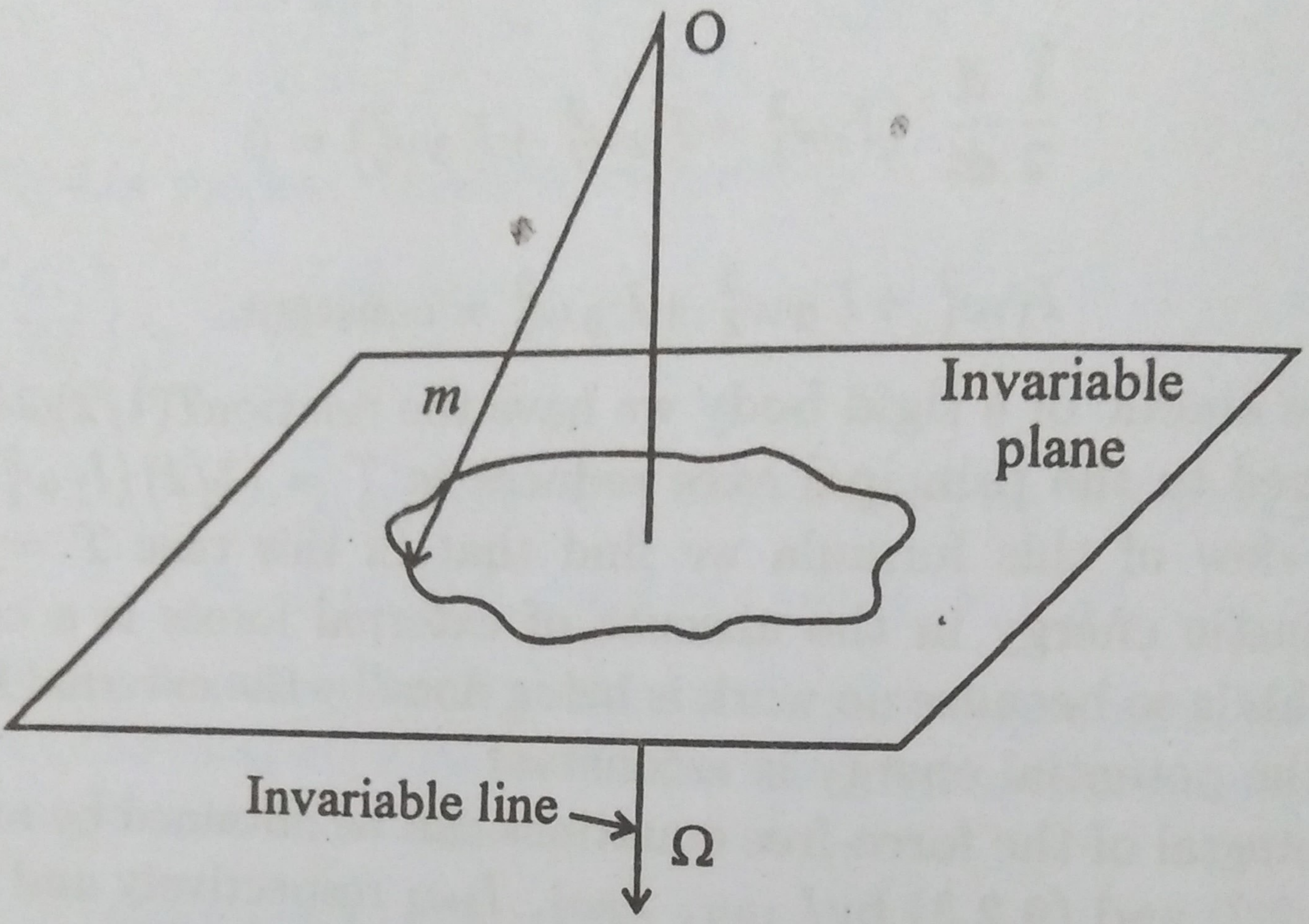
ot change with time. ne. For such a body we have proved that $T = (1/2)\omega\cdot$**L**is also a motion; i.e. $\vec{\omega}\cdot$**L**= constant. Since **L**is constant, it follow mponent of $\vec{\omega}$ along**L**, (or in other words, the projection of $\vec{\omega}$ so constant. The tip of the angular velocity vector $\vec{\omega}$ describ lled*invariable plane.* As the rigid body rotates, an observer fix



gure 9.1: A symmetrical top in which the torque or net moment out any point is zero.

the body coordinate axes would see*rotation*or*precession*of the an locity vector$\vec{\omega}$about the angular momentum vector**L**.

4 Torque-free Motion of a Symmetrical To

e consider the motion of a rigid body under the action of a set o ch that the net torque (i.e. net moment of forces) about an point dy is zero. The c.m. of such a body is either at rest or movin iform velocity. Without loss of generality we may suppose that t. Thus the problem reduces to that of rotation about a*fixed point* applied torque.

Euler's dynamical equations in the form (9.2.1 - 9.2.3) govern this motion. To further simplify simplify calculations we suppose that the rigid body is a symmetrical top. For any point on the axis of symmetry of such a top, the axis itself and two other axes perpendicular to each other and to it will be the principal axes there. We take the origin at the c.m. of the top so that body coordinate axes are coincident with the principal axes there. We choose the Z-axis along the axis of symmetry.

If I_1, I_2, I_3 are the principal moments of inertia at the origin (the c.m.) of the top, then by definition of a symmetrical top $I_2 = I_1$, and I_3 will be different from I_1.

Under these assumptions Euler's dynamical equations (8.2.1–8.2.3) become

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \tag{9.4.1}$$
$$I_1\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0 \tag{9.4.2}$$
$$I_1\dot{\omega}_3 = 0 \tag{9.4.3}$$

To make our treatment more general, we assume that the angular velocity $\vec{\omega}$ of the top is not aligned along any of the principal axes.

From equation (9.4.3), since $I_3 \neq 0$, $\dot{\omega}_3 = 0$ which gives

$$\omega_3 = \text{constant} \tag{9.4.4}$$

Now equations (9.4.1) and (9.4.2) may be rewritten as

$$\dot{\omega}_1 + \frac{I_3 - I_1}{I_1}\omega_2\omega_3 = 0 \tag{9.4.5}$$

and

$$\dot{\omega}_2 + \frac{I_3 - I_1}{I_1}\omega_3\omega_1 = 0 \tag{9.4.6}$$

Now let

$$\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3 = \text{a constant} = \gamma\omega_3 \tag{9.4.7}$$

Then (9.4.5) and (9.4.6) can be rewritten as

$$\dot{\omega}_1 + \Omega\omega_2 = 0 \tag{9.4.8}$$
$$\dot{\omega}_2 + \Omega\omega_1 = 0 \tag{9.4.9}$$

Equation (9.4.8) and (9.4.9) are coupled first order linear differential equations. To solve them, we multiply (9.4.9) by ι and add to(9.4.9). This gives

$$(\dot{\omega}_1 + \iota\dot{\omega}_2) + \Omega(\omega_2 - \iota\omega_1) = 0 \quad \text{or} \quad \frac{d}{dt}(\omega_1 + \iota\omega_2) + \Omega(-i^2\omega - \iota\omega_1) = 0$$

or

$$\frac{d}{dt}(\omega_1+\iota\omega_2)-\iota\Omega(\omega_1+\iota\omega_2) = 0 \qquad (9.4.10)$$

Let $\eta=\omega_1+\iota\omega_2$, then

$$\dot{\eta}= \dot{\omega}_1+\iota\dot{\omega}_2 \qquad (9.4.11)$$

and (9.4.10) reduces to

$$\frac{d\eta}{dt} - \iota\omega_1\eta = 0 \qquad (9.4.12)$$

whose general solution is given by

$$\eta = \eta_0 e^{\iota\Omega t} \qquad (9.4.13)$$

Using (9.4.11), we obtain from (9.4.13)

$$\omega_1 = \eta_0 \cos\Omega t, \quad \omega_2 = \eta_0 \sin\Omega t \qquad (9.4.14)$$

where η_0 is assumed to be an arbitrary *real* constant.

On squaring and adding the last two equations, we obtain

$$\omega_1^2+\omega_2^2 = \eta_0^2$$

which shows that the sum of squares of ω_1 and ω_2 is constant (see equation (9.4.4)). Also ω_3 is constant. Therefore

$$\omega_1^2+\omega_2^2+\omega_3^2 = \text{constant}$$

or

$$|\vec{\omega}| \equiv \omega = \sqrt{\omega_1^2+\omega_2^2+\omega_3^2} = \sqrt{\eta_0^2+\omega_3^2} = \text{constant}$$

Equations (9.4.14) are parametric equations of a circle and show that the terminal end of the angular velocity vector $\vec{\omega}$ traces a circle with time t in the OXY-plane. This implies that the vector $\vec{\omega}$ precesses in a cone about the OZ-axis, (the symmetry axis of the top) with constant angular frequency (= angular velocity) Ω, (see figure 9.2), while ω_3 remains constant around the symmetry axis.

As observed by an observer stationed in the body coordinate system (i.e. located in the top), the vector $\vec{\omega}$ traces out a cone around the body symmetry axis. This is called the *body cone* and in the body reference frame its semi-vertical angle is given by

$$\tan\phi_B = \frac{\sqrt{\omega_1^2+\omega_2}}{\omega_3} = \frac{\eta_0}{\omega_3}$$

The above discussion is based on the assumption that the rigid body is under no external force. From the view point of an observer in the fixed

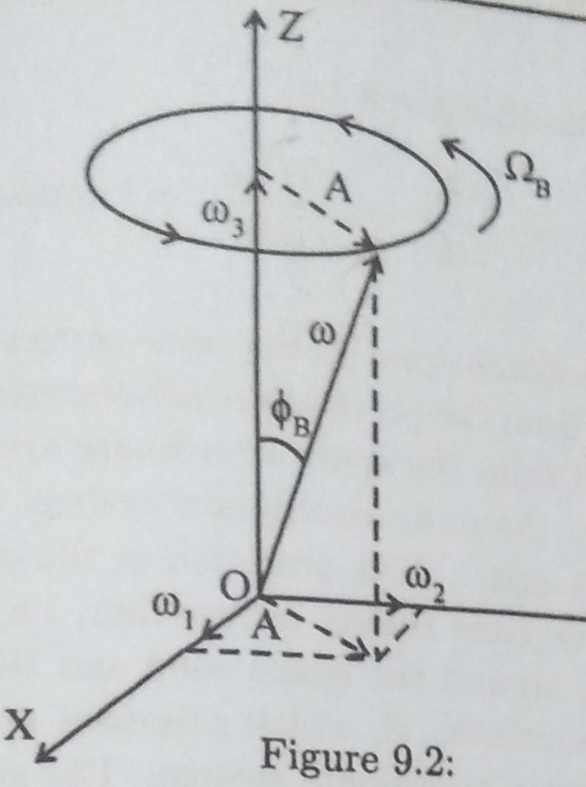


Figure 9.2:

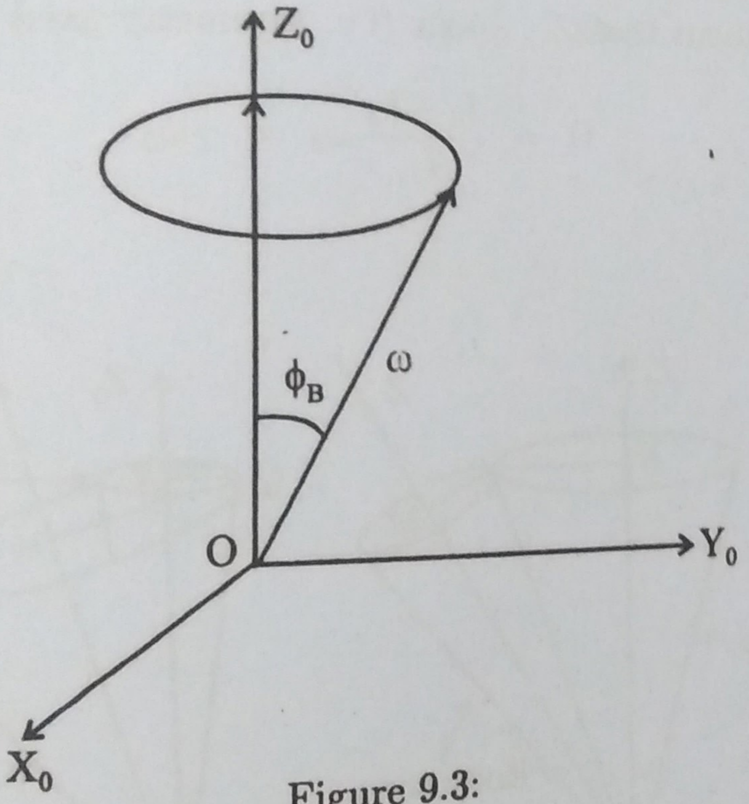


Figure 9.3:

(or space or inertial) coordinate system, there will be two constants of motion, the angular momentum and kinetic energy. i.e. $\mathbf{L}(t)$ = constant, and is fixed about theOZ $_0$-axis, as can be seen from figure 9.2.

Since the c.m. is fixed, the K.E. is all rotational and constant and therefore

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\cdot\mathbf{L} \quad = \text{constant}$$

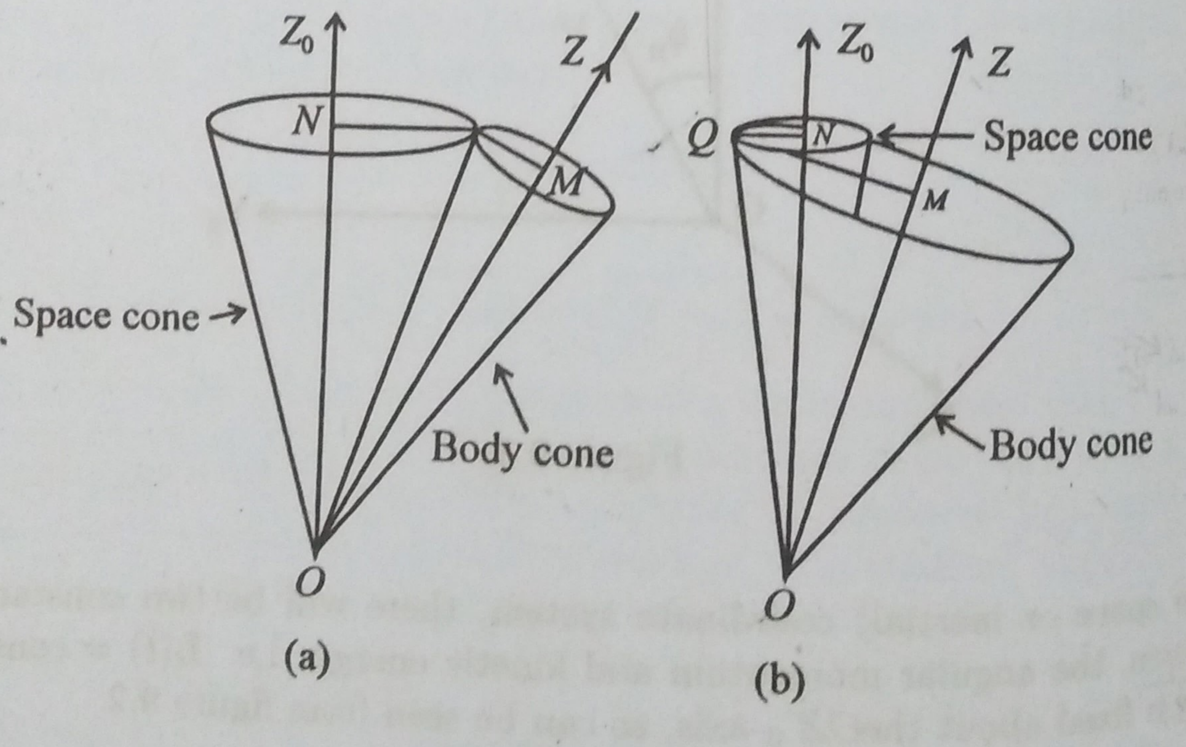
We know that **L** is constant; T_{rot} will also be constant if the tip of the angular velocity vector$\vec{\omega}$rotates (or*precesses*) in such that a way that its projection on the angular momentum vector**L**is constant. From figure 9.3

the angleϕ_L between$\vec{\omega}$and**L**is given by

$$\cos\phi_L = \frac{\vec{\omega}\cdot\mathbf{L}}{\omega L} = \frac{2T_{\mathrm{rot}}}{\omega L} = \text{constant}$$

Angleϕ_L remains constant and is the semi-vertical angle of the *space cone*. This cone results from the precession of$\vec{\omega}$about the constant angular momentum**L**, as viewed from the space coordinate system. On the other hand when viewed from the body coordinate system, $\vec{\omega}$ precesses around the OZ-axis (symmetry axis). This situation is shown in figure 9.4 and may be described as one cone rolling on another, i.e. the body cone is rolling without slipping around the space cone and the line of contact is direction of the angular velocity $\vec{\omega}$, which precesses around the OZ_0-axis when viewed from the space coordinate system. The angular frequency of precession of$\vec{\omega}$about theOZ_0-axis (i.e. symmetry axis) is given by

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3 = \gamma\omega_3$$



(a) (b)

Figure 9.4:

Similarly the angular frequency of precession of $\vec{\omega}$ about the OZ_0-axis is given by

$$\Omega_L = \gamma\omega_3\frac{\sin\phi_B}{\sin\phi_L}$$

Depending on the values of$I_1(I_x)$ and$I_3(I_z)$ the body cone may roll outside or inside the space, as shown in figure 9.**.

or

$$\mathbf{a}_f = \frac{d\mathbf{v}}{d} + \vec{\Omega}\times\mathbf{v} \qquad (9.5$$

where$\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$is the velocity of the mass centre (in the rotating coordinate system). Substituting for$\mathbf{a}_f=\mathbf{a}_c$ from (9.5.3) into (9.5.1), we obtain

$$M\left(\frac{d\mathbf{v}}{dt} + \vec{\Omega}\times\mathbf{v}\right) = \mathbf{F}$$

which is equivalent to

$$\left.\begin{aligned} M(\dot{v}_1+\Omega_2 v_3-\Omega_3 v_2) &= F_1\\ M(\dot{v}_2+\Omega_3 v_1-\Omega_1 v_3) &= F_2\\ M(\dot{v}_3+\Omega_1 v_2-\Omega_2 v_1) &= F_3\end{aligned}\right\} \qquad (9.5.4)$$

From (9.5.2), on using

$$\left(\frac{d\mathbf{L}}{dt}\right)_f = d\mathbf{L}dt + \Omega\times\mathbf{L}$$

and the relation $\mathbf{L}=I_1\omega_1\mathbf{i}_1+I_2\omega_2\mathbf{j}+I_3\omega_3\mathbf{k}$, we obtain the equations

$$\begin{aligned} I_1\dot{\omega}_1\mathbf{i} &+ I_2\dot{\omega}_2\mathbf{j}+I_3\dot{\omega}_3\mathbf{k} + (\Omega_2 L_3-\Omega_3 L_2)\mathbf{i}\\ &+ (\Omega_2 L_1-\Omega_1 L_3)\mathbf{j} + (\Omega_1 L_2-\Omega_2 L_1)\mathbf{k} = \mathbf{G}\end{aligned}$$

From this vector equation we obtain the following three scalar equations

$$I_1\dot{\omega}_1 + \Omega_2 L_3 - \Omega_3 L_2 = G_1$$

$$I_2\dot{\omega}_2 + \Omega_3 L_1 - \Omega_1 L_3 = G_2$$

and

$$I_3\dot{\omega}_3 + \Omega_1 L_2 - \Omega_2 L_1 = G_3$$

where we have used the results

$$L_1 = I_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3$$

we have

$$\left.\begin{aligned} I_1\dot{\omega}_1+\omega_3\Omega_2 I_3-\omega_2\Omega_3 I_2 &= G_1\\ I_2\dot{\omega}_2+\omega_1\Omega_3 I_1-\omega_3\Omega_1 I_3 &= G_2\\ I_3\dot{\omega}_3+\omega_2\Omega_1 I_2-\omega_1\Omega_2 I_1 &= G_3\end{aligned}\right\} \qquad (9.5.5)$$

which are the same as for a rigid body with a fixed point. In these equations I_1, I_2, I_3 denote principal moments of inertia at the centroid of the body. The sets of equations (9.5.4) and (9.5.5) constitute six equations for the components of velocity of the mass centre and the components of angular velocity of the body. For any of these six equations, we can substitute the law of conservation of energy, *viz.* $T+V=E$, provided the external forces are conservative. The last equation is equivalent to

$$\frac{1}{2}M(v_1^2+v_2^2+v_3^2) + \frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+V = E$$

9.6 Stability of Rigid Body Rotations

The problem of stability of rotations of a rigid body was first studied by Euler in 1749. We will assume that there is no external force on the rigid body and it is rotating about one of its principal axes. The motion of the rigid body will be deemed to be stable if under a small perturbation the body will return to its former state of motion or will perform small oscillations about the fixed point (or axis).

LetI_1, I_2, I_3 denote the principal moments of the body and without loss of generality we suppose that$I_1 < I_2 < I_3$. We choose the body coordinate system along the principal axes and take the body axisOX_1 corresponding to the principal moment I_1, as the axis of rotation. Then the angular velocity$\vec{\omega}$of the body can be represented as

$$\vec{\omega} = \omega_1 \mathbf{i} \tag{9.6.1}$$

When a small perturbation is applied, the axis of rotation is slightly displaced and the angular velocity then takes the form

$$\vec{\omega} = I_1 \mathbf{i}+\lambda\mathbf{j}+\mu\mathbf{k} \tag{9.6.2}$$

whereλ, $\mu$are very very small parameters. The Euler dynamical equations are

$$\begin{aligned} I_1 \dot{\omega}_1 -(I_2 - I_3)\omega_2\omega_3 &= 0 \\ I_2 \dot{\omega}_2 -(I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3 \dot{\omega}_3 -(I_1 - I_2)\omega_1\omega_2 &= 0 \end{aligned}$$

For the problem under discussion, from (9.6.2),$\omega_2 = \lambda$, $\omega_3 = \mu$and therefore the Euler equations become

$$\left.\begin{aligned} I_1 \dot{\omega}_1 + (I_2 - I_3)\lambda\mu &= 0 \\ I_2 \dot{\lambda} + (I_3 - I_1)\mu\omega_1 &= 0 \\ I_3 \dot{\mu} + (I_1 - I_2)\omega\lambda &= 0 \end{aligned}\right\} \tag{9.6.3}$$

Since the produce$\lambda\mu$is negligibly small, the first of equations (9.6.3) reduces to $\dot{\omega}_1 = 0$ orω_1 = constant. From the second and the third equations of (9.6.3) we obtain

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu \tag{9.6.4}$$

$$\dot{\mu} = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\lambda \tag{9.6.5}$$

where each term in the parentheses is constant. Differentiating each of the equations w.r.t. t and eliminating $\dot{\lambda}$ or $\dot{\mu}$ with the help of the other, we obtain the second order differential equations

$$\ddot{\lambda}+ \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2 \lambda = 0 \tag{9.6.6}$$

and

$$\ddot{\mu}+ \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2 \mu = 0 \tag{9.6.7}$$

Mathematically equation (9.6.7) is exactly the same as (9.6.6) with λ replaced byμ.

The solution of (9.6.7) is given by

$$\lambda(t) = A e^{\iota\Omega_1 t} + B e^{-\iota\Omega_1 t} \tag{9.6.8}$$

where

$$\Omega_1^2 = \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\omega_1^2 \tag{9.6.9}$$

From our assumption that $I_1 < I_2 < I_3$, it follows that Ω_1 is real. The solution (9.6.8) therefore represents oscillatory motion with a frequency Ω_1. The solution for$\mu(t)$ is the same as in (9.6.8) i.e.

$$\mu(t) = A e^{\iota\Omega_1 t} + B e^{-\iota\Omega_1 t} \tag{9.6.8$'$}$$

From (9.6.8) and (9.6.8′) we conclude that the small perturbative terms in (9.6.2) do not increase with time but oscillate around the equilibrium values$\lambda = 0$, and$\mu = 0$. Hence the rotation about theOX- axis is stable.

Similarly if we consider the rotations about theOYandOZ-axes, then the corresponding angular frequencies Ω_2 and Ω_3 can be obtained from (9.6.9) as

$$\Omega_2^2 = \frac{(I_2 - I_3)(I_2 - I_1)}{I_3 I_1}\omega_2^2 \tag{9.6.10}$$

and

$$\Omega_3^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\omega_3^2 \tag{9.6.11}$$

Since$I_1 < I_2 < I_3$ we find that Ω_1, Ω_3 are real whereas Ω_2 is pure imaginary.

If follows that if the rotation takes place around theOXorOZ- axis, the perturbation produces oscillatory motion and the rotation is stable. If the rotation takes place around the OY-axis, because of Ω_2 being imaginary, the exponential factor $e^{-\iota\Omega_2 t}$ in the solution for $\lambda(t)$ and $\mu(t)$ becomes which the passage of time and therefore motion is unstable.

above analysis is applicable to any rigid body which is rotating about one of the three principal axes (with distinct moments of inertia) at point fixed in it. Therefore we conclude that the rotation around the principal axis corresponding to either the largest or the smallest M.I. is stable and that around the principal axis with intermediate M.I. is unstable.

two of the M.I. say, I_1 and I_2 are equal, then from (9.6.5) we find that $= 0$) and therefore $\mu=\mu_0$, a constant. The equation (9.6.4) for λ now becomes

$$\frac{d\lambda}{dt} = \mu_1 \mu_0 \tag{9.6.12}$$

where $\mu_1 = (I_3 - I_1)/I_1$. The solution of (9.6.12) is $\lambda=\mu_1 \mu_0 t+\mu_a$.

This shows that the perturbation will increase linearly with time and the rotation around the OX- axis is unstable. We can obtain a similar result for rotation about the OY-axis. It can be further shown that the motion will be stable only when the rigid body is rotating about the OZ-axis irrespective of whether I_3 is greater than or less than $I_1 = I_2$.

9.7 Euler's Angles and Rigid Body Motion

A rigid body constrained to rotate about a fixed point has only three degrees of freedom. Therefore we require three parameters to specify the configuration of such a body. Euler's angle are three angular coordinates which are used to specify the configuration (orientation) of a rigid body. The Euler angles are usually denoted by θ, ϕ, ψ. Note that there is no universally agreed notation, neither is there agreed convention about their signs.

Let the fixed point about which the body is rotating be O. To define the Euler angles we consider a coordinate system (or a frame of reference) $OX_0Y_0Z_0$ fixed in space, and another coordinate system $OXYZ$ fixed in the body and rotating with it. The first coordinate system is usually referred to as *space* or *fixed* or *inertial* coordinate system, whereas the second coordinate system is referred to as *body* or *moving* or *rotating* coordinate system. We suppose that the two coordinate systems are initially (i.e. at $t = 0$) coincident and define the the Eulerian angles θ, ϕ, ψ in relation to the coordinate system, as follows.