Chapter 9

Motion of a Rigid Body in Space

The plane motion of a rigid body discussed in chapter 8 is quite simple. Here the axis of rotation is fixed and therefore its direction does not change. In the case of general motion of a rigid body the direction of the axis of rotation is not fixed. Consequently the situation is much more complicated; even in the case of a body on which no forces are acting, the problem is not simple.

In this chapter we will discuss the motion of a rigid body in space. First we will discuss the motion of such a body when it is fixed about a point. Later we discuss the general motion in which both translation and rotation are involved.

We will also discuss the stability of motion and other related problems.

9.1 Euler's Dynamical Equations

Let a rigid body be rotating with angular velocity $\vec{\omega}$ about a point O fixed both in space and in the body. Let OX, OY, OZ be principal axes at O. If $\mathbf{L}$, $[I_{ij}]$, $\vec{\omega}$ be angular momentum, M.I. matrix and angular velocity at O, then

$$[\mathbf{L}] = [\mathbf{I}]\,[\vec{\omega}]$$

or

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Since the axes OX, OY, OZ are principal axes, $I_{xy} = I_{xz} = I_{yz} = 0$ therefore we can write

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \omega_x I_{xx} \\ \omega_y I_{yy} \\ \omega_z I_{zz} \end{bmatrix}$$

or

$$\begin{aligned} \mathbf{L} &= \omega_x I_x \mathbf{i} + \omega_y I_y + \omega_z I_z \\ &= \omega_1 I_1 \mathbf{i} + \omega_2 I_2 \mathbf{j} + \omega_3 I_3 \mathbf{k} \end{aligned}$$

where $(I_1, I_2, I_3) \equiv (I_x, I_y, I_z)$ are principal moments.

Now the rate of change of any vector function $\mathbf{F}$ in fixed and rotating coordinate systems is related by

$$\left(\frac{d\mathbf{F}}{dt}\right)_f = \left(\frac{d\mathbf{F}}{dt}\right)_r + \vec{\omega} \times \mathbf{F}$$

Replacing $\mathbf{F}$ by $\mathbf{L}$ in the last equation we have

$$\begin{aligned} \left(\frac{d\mathbf{L}}{dt}\right)_f &= \left(\frac{d\mathbf{L}}{dt}\right)_r + \vec{\omega} \times \mathbf{L} \\ &= \frac{d}{dt}(I_1 \omega_1 \mathbf{i} + I_2 \omega_2 \mathbf{j} + I_3 \omega_3 \mathbf{k}) + \vec{\omega} \times \mathbf{L} \\ &= I_1 \dot{\omega}_1 \mathbf{i} + I_2 \dot{\omega}_2 \mathbf{j} + I_3 \dot{\omega}_3 \mathbf{k} + \vec{\omega} \times \mathbf{L} \end{aligned}$$

where the symbols on the R.H.S. refer to the rotating coordinate system. But $d\mathbf{L}/dt = \mathbf{G}$, the total external torque (in the fixed or inertial coordinate system). Therefore on substitution

$$\mathbf{G} = I_1 \dot{\omega}_1 \mathbf{i} + I_2 \dot{\omega}_2 \mathbf{j} + I_3 \dot{\omega}_3 \mathbf{k} + \vec{\omega} \times \mathbf{L}$$

This vector equation is equivalent to the following three scalar equations:

$$\begin{aligned} G_x &= I_1 \dot{\omega}_1 + \omega_2 L_3 - \omega_3 L_2 \\ G_y &= I_2 \dot{\omega}_2 + \omega_3 L_1 - \omega_1 L_3 \\ G_z &= I_3 \dot{\omega}_3 + \omega_1 L_2 - \omega_2 L_1 \end{aligned}$$

Now using the results $L_1 = \omega_1 I_1$, $L_2 = \omega_2 I_2$, $L_3 = \omega_3 I_3$, which are true w.r.t. principal axes, we have

$$\begin{aligned} G_x &= I_1 \dot{\omega}_1 + (I_3 - I_1)\omega_2 \omega_3 \\ G_y &= I_2 \dot{\omega}_2 + (I_1 - I_2)\omega_3 \omega_1 \\ G_z &= I_3 \dot{\omega}_3 + (I_2 - I_3)\omega_1 \omega_2 \end{aligned} \tag{9.1.1}$$

These equations are called *Euler's dynamical equations*. They describe the motion of a rigid body fixed at a point.

9.2 Deductions from Euler's Equations

Here we will discuss some results which follow directly from the Euler dynamical equations. In the absence of external forces, **G**= 0, and the Euler equations (9.1.1) reduce to

$$I_1 \dot{\omega}_1 + (I_3 - I_1)\omega_2 \omega_3 = 0 \quad (9.2.1)$$

$$I_2 \dot{\omega}_2 + (I_1 - I_2)\omega_3 \omega_1 = 0 \quad (9.2.2)$$

$$I_3 \dot{\omega}_3 + (I_2 - I_3)\omega_1 \omega_2 = 0 \quad (9.2.3)$$

Multiplying (9.2.1), (9.2.2) and (9.2.3) by ω_1, ω_2, ω_3 respectively and adding, we have

$$I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = 0$$

or

$$I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 = \text{constant}$$

Now for the kinetic of a rigid body we have the relation$T(1/2)\vec{\omega}\cdot$**L**which when referred to the principal axes reduces to $T = (1/2)\,(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$. In view of this formula we find that in this case T = constant. i.e. the kinetic energy in the absence of external forces is a constant of motion. This is so because no work is being done by the external forces, and therefore the potential energy is a constant. (9.1.1)

Another integral of the force-free equations can be obtained by multiplying (9.2.1), (9.2.2) and (9.2.3) by$I_1\omega_1$, $I_2\omega_2$, $I_3\omega_3$ respectively and adding

$$I_1^2 \omega_1 \dot{\omega}_1 + I_2^2 \omega_2 \dot{\omega}_2 + I_3^2 \omega_3 \dot{\omega}_3 = 0$$

or

$$\frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right) = 0$$

or

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{constant} \quad (9.2.4)$$

Now the angular momentum**L**w.r.t. the triad of principal axes is given by

$$\mathbf{L} = I_1\omega_1\mathbf{i} + I_2\omega_2\mathbf{j} + I_3\omega_3\mathbf{k}$$

Therefore

$$\mathbf{L}^2 \equiv |\mathbf{L}|^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

(9.1.1) Hence (9.2.4) expresses the fact that the magnitude of**L**is a constant of motion. Physically this can be related to the absence of the external torque about the point$O$.