$$\lambda_3 = 61\alpha, \quad Z = \frac{1}{\sqrt{10}}\begin{bmatrix} \\ 0 \\ -3 \end{bmatrix} \equiv \frac{1}{\sqrt{10}}(\mathrm{i}-3\mathrm{k})$$

## 7.4 Equimomental Systems

Two distributions of matter are said to be *equimomental* if they
same moment of inertia about any line in space. Such system
esting because two equimomental systems will have the same
behaviour, i.e. behave in the same way under identical forces.

### 7.4.1 Necessary and sufficient conditions

**Theorem**

Two systems $S_1$ and $S_2$ are equimomental if and only if

(i) they have the same mass.

(ii) they have the same centroid.

(iii) they have the same principal axes and principal moments at th
of mass.

**Proof**

*The condition is sufficient*

Here we will prove that if conditions (i), (ii) and (iii) are satisfied, tl
systems will be equimomental.

Let these conditions be satisfied. Let $C$ and $M$ be the common ce
and mass of the two systems and let $I_1$, $I_2$, $I_3$ be the principal mo
of inertia w.r.t. principal axes at $G$.

Let $l$ be an arbitrary line in space. Through $C$ we draw a line $l_0$ pa
to $l$. Then the moment of inertia of each system about $l_0$ is given by

$$I_0 = I_1\lambda + I_2\mu + I_3\nu$$

...line $l$, which will be ...moment of inertia of each system about

$$I_l = I_0 + Md^2 = I_1\lambda + I_2\mu + I_3\nu + Md^2$$

...nce the moment of inertia of both the systems about an arbitrary line is ...e same, it follows that the systems are equimomental. Hence the condi...on is sufficient.

*...he condition is necessary*

...Now we assume that the two systems are equimomental and then deduce

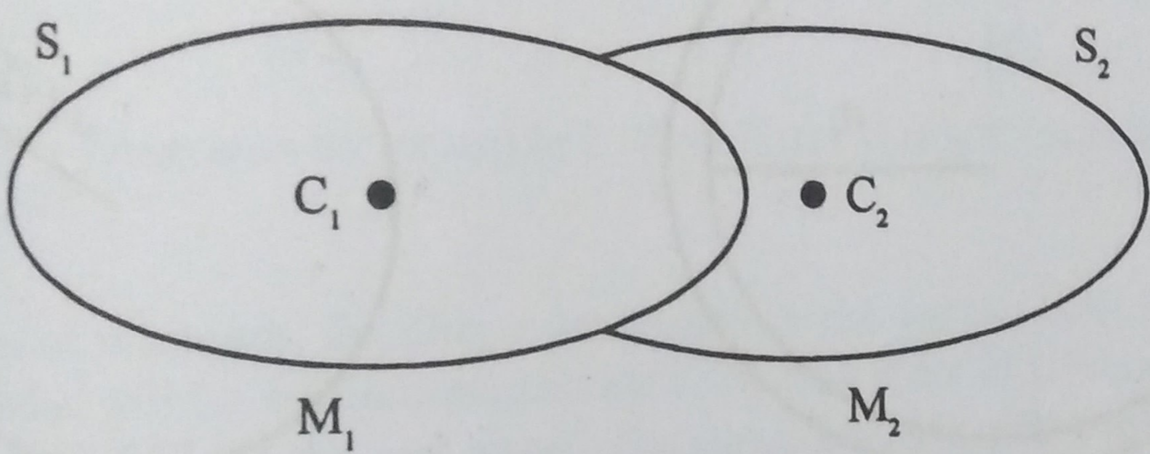


Figure 7.5: Two equimomental bodies.

that the conditions (i), (ii) and (iii) are satisfied.

Let $M_1$ and $M_2$ be the masses of the systems, $S_1$ and $S_2$ and $C_1$ and $C_2$ be the centroids respectively. Since the systems are equimomental, the moment of inertia $I$ of each system is the same about any line; in particular about the line $\overline{C_1C_2}$.

Let $J$ be the M.I. of each system about a line $l'$ through $C_1$ and perpendicular to the line $\overline{C_1C_2}$.

Then the M.I. of the system $S_1$ about a line $l''$ parallel to $l'$ and through the point $C_2$ is given by $I_1 = J + M(\overline{C_1C_2})^2$. And the M.I. of the system $S_2$ about the line $l''$ is given by $I_2 = J - M(\overline{C_1C_2})^2$

Since $S_1$ and $S_2$ are equimomental, we must have

$$J + M(\overline{C_1C_2})^2 = J - M(\overline{C_1C_2})^2 2M(\overline{C_1C_2})^2 = 0 \Rightarrow \overline{C_1C_2} = 0$$

This means that $C_1 \equiv C_2 \equiv C$ i.e. the two systems have the same centroid.

(iii) The two systems have the same centroid $C$, therefore, they have the same momental ellipsoid at $C$.

Therefore they have the same principal moments of inertia and axes of inertia at $C$. So condition (iii) is satisfied. Hence the theorem.

## Example 1

Show that a hoop of mass $m$ and radius $a/\sqrt{2}$ is equimomental with a circular plate of mass $m$ and radius $a$.

## Solution

The M.I. of a circular disc (or plate) of mass $m$ and radius $a$ about an axis

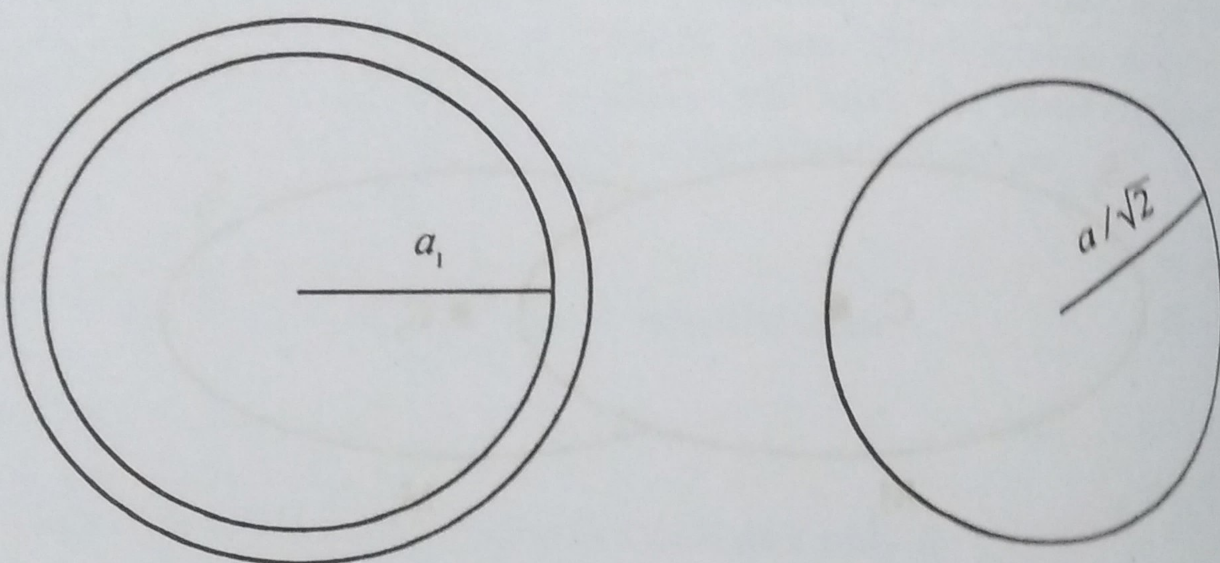


Figure 7.6: A hoop and a circular plate.

through its centre and perpendicular to its plane is given by

$$I_1 = \frac{1}{2} m a^2$$

The M.I. of a hoop of mass $m$ and radius $b$ about an axis through its centre and perpendicular to its plane is given by

$$I_2 = m b^2$$

The two systems will be equimomental if

$$I_1 = I_2 \text{ i.e. } \frac{1}{2} m a^2 = m b^2 \text{ i.e. } b = a/\sqrt{2}$$

It follows that the two systems are equimomental.

## Example 2

Find an equimomental system of particles for a uniform rod $AB$ of mass $M$.

Solution

Let $O$ be the centre of mass of the rod. If we replace the rod by three

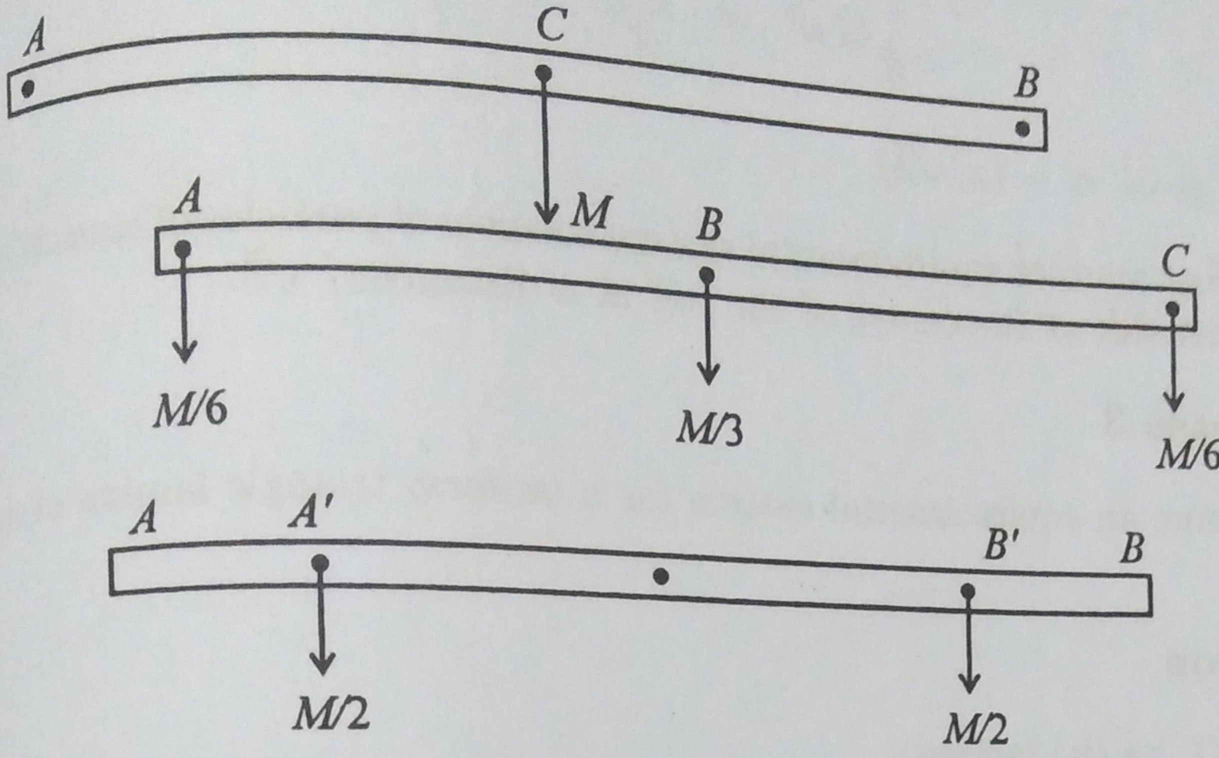


Figure 7.7: Diagrams for example 2: A uniform rod with masses at different positions.

particles of masses $m$, $M-2m$, $m$ as shown in the figure, then the system of particles will be equimomental with the rod if its M.I. about any line is equal to the M.I. of the rod about the same line. We will take the moment of inertia about a perpendicular line through the centre $O$ of the rod. We take the length of the rod as $2a$.

Total $M.I.$ of the three particles is given by

$$I_1 = ma^2 + 0 + ma^2 = 2ma^2$$

If $I_2$ denotes the M.I. of the rod about a perpendicular axis through its centre, then

$$I_2 = \frac{1}{3}Ma^2$$

The two systems will be equimomental if

$$I_1 = I_2 \quad i.e. \quad 2ma^2 = \frac{1}{3}Ma^2$$

which gives $m = M/6$.

Hence an equimomental system is given by masses $M/6$ at $A$, $(2/3)M$ at the centre and $M/6$ at $B$.

An alternative equimomental system can be found as follows.

Suppose we take two particles of masses $M/2$, $M/2$ at $A'$ and $B'$ with $OA' = a' = OB'$. Then this system will be equimomental with the rod if

$$\frac{1}{3}Ma^2 = \frac{M}{2}a'^2 + \frac{M}{2}a'^2$$

which gives $a' = (a/\sqrt{3})$.

Therefore another equimomental system consists of particles of mass $M/2$ each on either side of the centre of the rod at a distance $a/\sqrt{3}$.

### Example 3

Determine an equimomental system for a uniform triangle lamina of mass $M$.

### Solution

Let $ABC$ be the lamina.

We have already obtained the formula for its moment of inertia about any

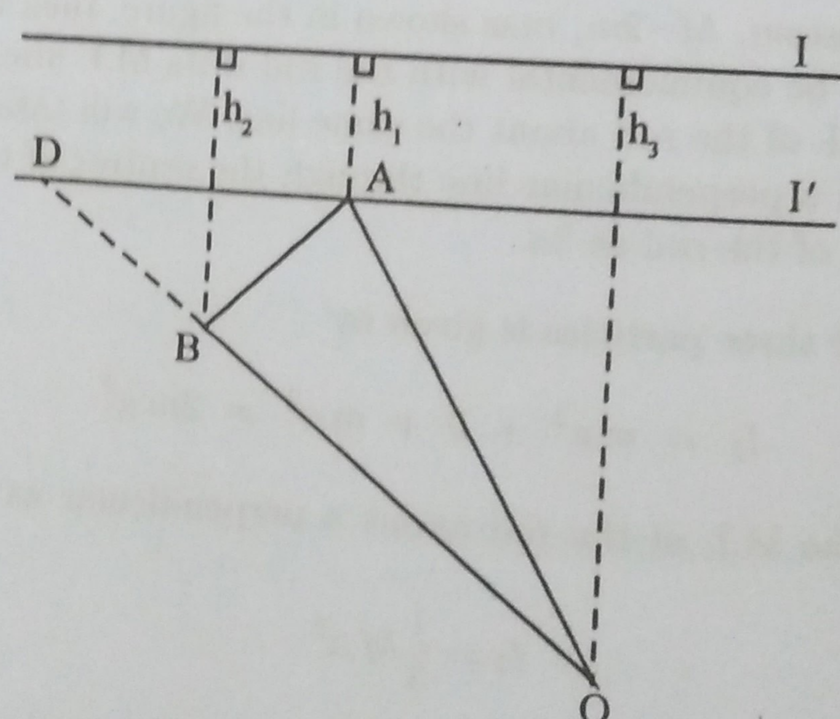


Figure 7.8: A uniform triangular lamina

side. Using this result we now find the $M.I.$ of the lamina about any line $l$ in the plane of the triangle $ABC$.

Let $h_1$, $h_2$, $h_3$ be the distances of the vertices $A$, $B$, $C$ respectively from the given line. We suppose that $h_1 < h_2 < h_3$. Through $A$ we draw a line $l'$ parallel to $l$ such that the line segment $CB$, when produced, meets it at the point $D$.

The distances ... will be 0, $h_2 - h_1$, $h_3 - h_1$ respectively.

If $M_1$ and $M_2$ denote the masses of the triangular regions $ACD$ and $ABD$, then

$$M = \text{mass of } ABC = M_1 - M_2$$

But

$$\frac{M_1}{M_2} = \frac{(1/2)\,(AD \times \text{height} \times \rho)}{(1/2)(AD \times \text{height} \times \rho} = \frac{h_3 - h_1}{h_2 - h_1}$$

Therefore

$$\frac{M_1}{M_2} - 1 = \frac{h_3 - h_1}{h_2 - h_1} - 1$$

$$\frac{M_1 - M_2}{M_2} = \frac{h_3 - h_1 - h_2 + h_1}{h_2 - h_1}$$

$$\frac{M}{M_2} = \frac{h_3 - h_2}{h_2 - h_1}$$

which gives

$$M_2 = \frac{M(h_2 - h_1)}{h_3 - h_2}, \qquad (M = M_1 - M_2)$$

and

$$M_1 = M + M_2 = M + \left(1 + \frac{h_2 - h_1}{h_3 - h_2}\right)$$

$$= M\left(\frac{h_3 - h_1}{h_3 - h_2}\right)$$

Moment of inertia of the triangular lamina $ABC$ about the line $l'$ is the difference of moments of inertia of the triangular laminas $ACD$ and $ABD$ about the same line. Denoting the same by $I_l'$, we have

$$I_l' = \frac{1}{6}M_1(h_3 - h_1)^2 - \frac{1}{6}M_2(h_2 - h_1)^2$$

Putting the values of $M_1$ and $M_2$, we obtain

$$I_l' = \frac{1}{6}M\frac{h_3 - h_1}{h_3 - h_2}(h_3 - h_1)^2 - \frac{1}{6}M\frac{h_2 - h_1}{h_3 - h_2}(h_2 - h_1)^2$$

$$= \frac{M}{6(h_3 - h_2)}\left[(h_3 - h_1)(h_3 - h_1)^2 - (h_2 - h_1)(h_2 - h_1)^2\right]$$

$$= \frac{M}{6(h_3 - h_2)}\left[(h_3 - h_1)^3 - (h_2 - h_1)^3\right]$$

Now using the formula

Suppose we take two particles of masses $M/2$, $M/2$ at $A'$ and $B'$ with $OA' = a' = OB'$. Then this system will be equimomental with the rod if

$$\frac{1}{3}Ma^2 = \frac{M}{2}a'^2 + \frac{M}{2}a'^2$$

which gives $a' = (a/\sqrt{3})$.

Therefore another equimomental system consists of particles of mass $M/2$ each on either side of the centre of the rod at a distance $a/\sqrt{3}$.

## Example 3

Determine an equimomental system for a uniform triangle lamina of mass $M$.

## Solution

Let $ABC$ be the lamina.

We have already obtained the formula for its moment of inertia about any side. Using this result we now find the $M.I.$ of the lamina about any line $l$ in the plane of the triangle $ABC$.

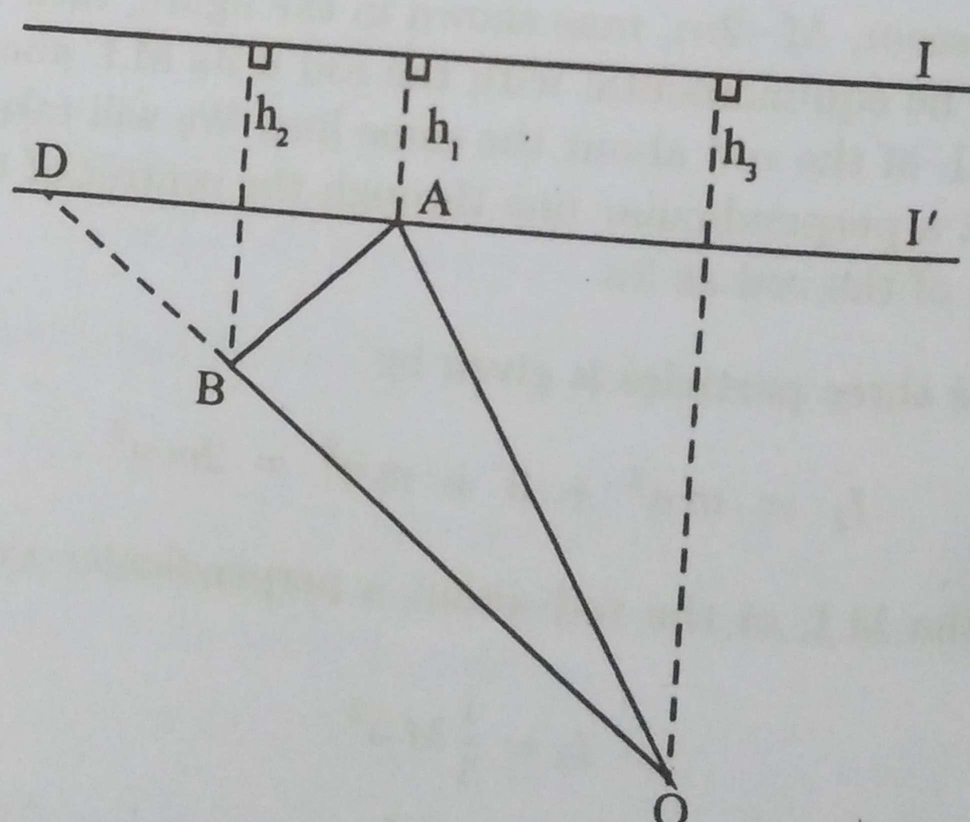


Figure 7.8: A uniform triangular lamina

Let $h_1$, $h_2$, $h_3$ be the distances of the vertices $A$, $B$, $C$ respectively from the given line. We suppose that $h_1 < h_2 < h_3$. Through $A$ we draw a line $l'$ parallel to $l$ such that the line segment $CB$, when produced, meets it at the point $D$.

mass of $ABC = M_1 - M_2$

$$\frac{M_1}{M_2} = \frac{(1/2)\,(AD\times \text{height}\times\rho)}{(1/2)(AD\times \text{height}\times\rho} = \frac{h_3 - h_1}{h_2 - h_1}$$

efore

$$\frac{M_1}{M_2} - 1 = \frac{h_3 - h_1}{h_2 - h_1} - 1$$

$$\frac{M_1 - M_2}{M_2} = \frac{h_3 - h_1 - h_2 + h_1}{h_2 - h_1}$$

$$\frac{M}{M_2} = \frac{h_3 - h_2}{h_2 - h_1}$$

ch gives

$$M_2 = \frac{M(h_2 - h_1)}{h_3 - h_2}, \qquad (M = M_1 - M_2)$$

$$M_1 = M + M_2 = M + \left(1 + \frac{h_2 - h_1}{h_3 - h_2}\right)$$

$$= M\left(\frac{h_3 - h_1}{h_3 - h_2}\right)$$

ment of inertia of the triangular lamina $ABC$ about the line $l'$ is the erence of moments of inertia of the triangular laminas $ACD$ and $ABD$ ut the same line. Denoting the same by $I_l'$, we have

$$I_l' = \frac{1}{6}M_1(h_3 - h_1)^2 - \frac{1}{6}M_2(h_2 - h_1)^2$$

ting the values of $M_1$ and $M_2$, we obtain

$$I_l' = \frac{1}{6}M\frac{h_3 - h_1}{h_3 - h_2}(h_3 - h_1)^2 - \frac{1}{6}M\frac{h_2 - h_1}{h_3 - h_2}(h_2 - h_1)^2$$

$$= \frac{M}{6(h_3 - h_2)}\left[(h_3 - h_1)(h_3 - h_1)^2 - (h_2 - h_1)(h_2 - h_1)^2\right]$$

$$= \frac{M}{6(h_3 - h_2)}\left[(h_3 - h_1)^3 - (h_2 - h_1)^3\right]$$

w using th

$$I_l' = \frac{M}{6(h_3 - h_2)}(h_3 - h_1 - h_2 + h_1)\times$$
$$\times \{(h_3 - h_1)^2 - (h_3 - h_1)(h_2 - h_1)(h_2 - h_1)^2\}$$
$$= \frac{M}{6(h_3 - h_2)}[(h_3 - h_2)(h_3^2 + h_1^2 - 2h_1h_3 + h_3h_2 - h_1h_3$$
$$- h_1h_2 + h_1^2 + h_2^2 + h_1^2 - 2h_1h_2)]$$
$$= \frac{M}{6}(3h_1^2 + h_3^2 + h_3h_2 - 3h_1h_3 - 3h_1h_3 - 3h_1h_2)$$

Now we will use the parallel-axis theorem to connect $I_l'$ with $I_0$, the the lamina about a parallel line through the centroid. In the usual no

$$I_l' = I_0 + Md^2$$

where $d$, the distance of c.m. from $l'$ is given by

$$d = \frac{1}{3}(\text{distance of } A + \text{distance of } B + \text{distance of } C)$$
$$= \frac{1}{3}(0 + (h_2 - h_1) + (h_3 - h_1))$$
$$= \frac{1}{3}(h_2 - h_1 + h_3 - h_1) = \frac{1}{3}(h_2 + h_3 - 2h_1) \quad (2$$

From (1) and (2)

$$\frac{M}{6}[3h_1^2 + h_3^2 + h_3h_2 - 3h_1h_3 - 3h_1h_3 - 3h_1h_2] = I_0 + M\frac{1}{a}(h_2 + h_3 - 2$$

Therefore

$$I_0 = \frac{M}{6}(3h_1^2 + h_3^2 + h_2h_3 - 3h_1h_3 - 3h_1h_2)$$
$$- \frac{M}{9}(h_2^2 + h_3^2 + 4h_1^2 + 2h_2h_3 - 4h_3h_1 - 4h_1h_2) \quad ($$

Using the parallel theorem again to connect $I_0$ and $I_l$, we have

$$I_l = I_0 + Md'^2$$

where $d'$ = distance of c.m. from $l$, and $d' = (h_1 + h_2 + h_3)/3$. On subst

$$I_l = \frac{M}{6}\left[3h_1^2 + h_3^2 + h_2h_3 - 3h_1h_3 - 3h_1h_2\right]$$
$$-\frac{M}{9}\left[h_2^2 + h_3^2 + 4h_1^2 + 2h_2h_3 - 4h_3h_1 - 4h_1h_2\right]$$
$$+\frac{M}{9}\left[h_1^2 + h_2^2 + h_3^2 + 2h_1h_2 + 2h_2h_3 2h_1h_3\right]$$
$$= \frac{M}{18}\left[9h_1^2 + 3h_3^2 + 3h_2^2 + 3h_2h_3 - 9h_1h_3 - 9h_1h_2\right.$$
$$-2h_2^2 - 2h_3^2 - 8h_1^2 - 4h_2h_3 + 8h_1h_3 + 8h_1h_2$$
$$\left.+2h_1^2 + 2h_2^2 + 2h_3^2 + 4h_1h_2 + 4h_2h_3 + 4h_1h_3\right]$$
$$= \frac{M}{18}\left(3h_1^2 + 3h_2^2 + 3h_3^2 + 3h_2h_3 + 3h_1h_2 + 3h_2h_3\right)$$
$$= \frac{M}{6}\left(h_1^2 + h_2^2 + h_3^2 + h_1h_3 + h_1h_3 + h_2h_3 + h_1h_2\right)$$

obtain an equimomental system, we write the last result as

$$= \frac{M}{12}\left(2h_1^2 + 2h_2^2 + 2h_3^2 + 2h_1h_3 + 2h_2h_3 + 2h_1h_2\right)$$
$$= \frac{M}{12}\left[(h_1^2 + h_2^2 + 2h_1h_2) + (h_2^2 + h_3^2 + 2h_2h_3) + (h_3^2 + h_1^2 + 2h_1h_3)\right]$$
$$= \frac{M}{12}\left[(h_1 + h_2)^2 + (h_2 + h_3)^2 + (h_1 + h_3)^2\right]$$
$$= \frac{M}{3}\left[\left(\frac{h_1 + h_2}{2}\right)^2 + \left(\frac{h_2 + h_3}{2}\right)^2 + \left(\frac{h_1 + h_3}{2}\right)^2\right]$$

here the terms on R.H.S. can be interpreted as moments of inertia due to articles of mass $(M/3)$ each at distances $(h_1 + h_2)/2$, $(h_2 + h_3)/2$, $(h_1 + h_3)/2$ from the line $l$. This shows that the equipmomental system consists three particles each of mass $M/3$ placed at the midpoints of the sides of the triangle $ABC$.

## Example 4

Show that a uniform solid cuboid of mass $M$ is equimomental with

(i) masses $(1/24)M$ at midpoints of its edges and $(1/2)M$ at its centre.

(ii) with masses $(1/24)M$ at its corners and $(2/3)M$ at its centre.

## Solution

The $M.I$s of a rectangular parallelepiped of sides $a$, $b$, $c$ about axes through the c.m. are

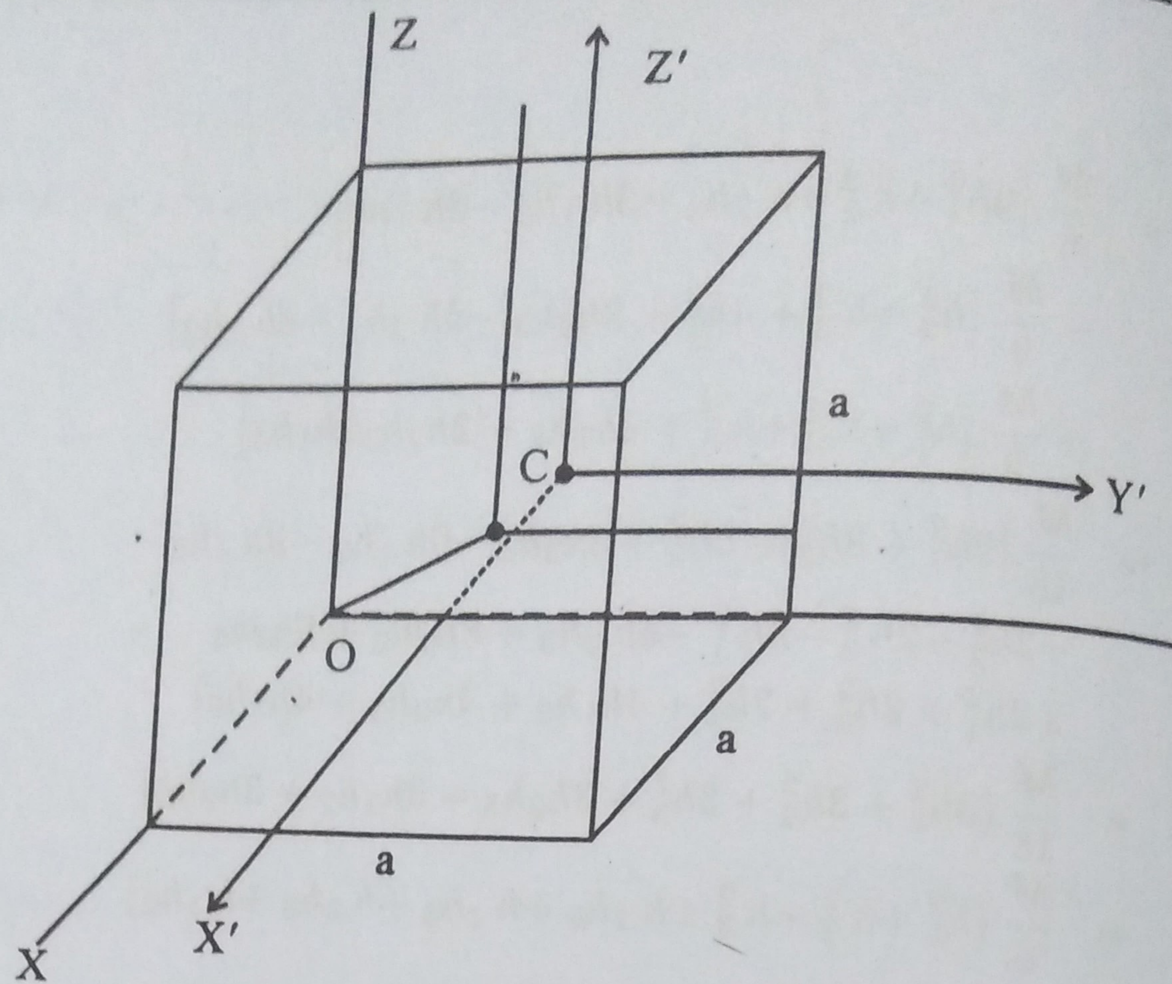


Figure 7.9: A solid cuboid

$$I_{11} = M(b^2 + c^2)/12, \quad I_{22} = M(a^2 + c^2)/12, \quad I_{33} = M(a^2 + b^2)$$

For a cuboid $a=b=c$, and therefore

$$I_{11} = I_{22} = I_{33} = Ma^2/6$$

(i) There are 12 edges of a cuboid. When masses $(M/24)$ are placed a midpoints of these edges and a mass $(M/2)$ at the centre, the total of the system will be $(12/24)M + (1/2)M = M$.

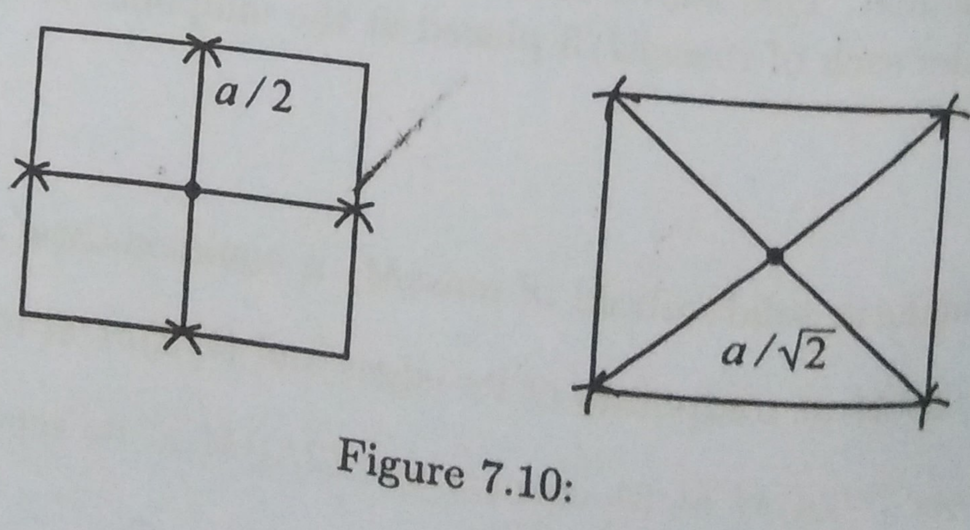


Figure 7.10:

Now if the moments of inertia of these masses about the coordinate axes through the c.m. are equal to $I_1 = I_2 = I_3 = Ma^2/6$, then the system will be equimomental.

...axis is ... distance of each of the 8 particles ... each of the other 4 particles is $a/\sqrt{2}$. M.I. of the particles at the midpoints about the $Z$-axis will be

$$I' = 8\times\frac{1}{24}M\times\left(\frac{a}{2}\right)^2 + 4\times\frac{M}{24}\times\left(\frac{a}{\sqrt{2}}\right)^2$$
$$= \frac{1}{3}\frac{Ma^2}{4}+\frac{1}{6}\frac{Ma^2}{2} = \frac{1}{6}Ma^2$$

...is equal to $I_{33}$. Hence the system is equimomental.

Total mass of the system $= 8\times(1/24)M + (2/3)M = M$. The ...ance of each mass from the $Z$-axis is $a/\sqrt{2}$. Hence the M.I. about the ...is of the whole system of particles will be

$$I'' = 8\times\frac{1}{24}M\times\left(\frac{a}{\sqrt{2}}\right)^2$$
$$= \frac{1}{3}\times M\times\frac{a^2}{2} = \frac{1}{6}Ma^2 \equiv I_{33}$$

...es the result.

## Questions and Problems

Four particles of masses $m$, $2m$, $3m$, $4m$ are located at the points ...$a$), $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$ respectively, and ...idly connected to one another by a light framework. Show that the ...pal moments of inertia of the system at the origin are $20ma^2$, $2(10+$ ...$a^2$, $2(10-\sqrt{5})ma^2$.

A square of side $a$ has particles of masses $m$,$2m$,$3m$,$4m$, at its ...s. Show that the principal moments of inertia at the centre of the ... are $2ma^2$,$3ma^2$,$5ma^2$ and find the directions of the principal axes.

...e uniform rods $OA$, $OB$, and $OC$ are each of unit length and unit ... Relative to a coordinate system $OXYZ$, the coordinates of $A$, $B$, and ...espectively $(1, 0, 0)$, $(0, 0, 1)$ and $(\sqrt{3})/2$, $1/2$, $0)$. Show that the ...al moments of inertia of the system at $O$ are $2/3$, $2/3+1/(2\sqrt{3})$ .../3$+1/(2\sqrt{3})$.

...ee particle system consists of masses and coordinates given as fol-