Next we find moment of inertia about y-axis. P3

$$I_{yy} = \Sigma (x_i^2 + z_i^2) m_i$$

$$= \int_V (x^2+z^2) \rho \, dV$$

$$= \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a (r^2\cos^2\varphi + z^2) \rho \, r \, dr \, d\varphi \, dz$$

$$= \rho \iiint (r^3\cos^2\varphi + r z^2) \, dr \, d\varphi \, dz$$

$$= \rho \int_{-h/2}^{h/2}\int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\varphi + \frac{r^2}{2} z^2\right)\Big|_0^a d\varphi \, dz$$

$$= \rho \iint \left(\frac{a^4}{4}\cos^2\varphi + \frac{a^2}{2} z^2\right) dz \, d\varphi$$

$$= \rho \int_0^{2\pi} \left(\frac{a^4}{4}\cos^2\varphi \, z\Big|_{-h/2}^{h/2} + \frac{a^2}{2}\frac{z^3}{3}\Big|_{-h/2}^{h/2}\right) d\varphi$$

$$= \rho \int \left(\frac{a^4}{4}\cos^2\varphi\,(h) + \frac{a^2}{6}\left(\frac{2h^3}{8}\right)\right) d\varphi$$

$$= \rho \int \left(\frac{h a^4}{4}\cos^2\varphi + \frac{a^2 h^3}{24}\right) d\varphi$$

$$= \rho \int_0^{2\pi} \frac{h a^4}{8}(2\cos^2\varphi)\, d\varphi + \frac{\rho a^2 h^3}{24}\int_0^{2\pi} d\varphi$$

$$= \rho \int_0^{2\pi} \frac{h a^4}{8}(1+\cos 2\varphi)\, d\varphi + \frac{\rho a^2 h^3}{24}(2\pi)$$

$$= \rho \frac{h a^4}{8} \left. \varphi + \frac{\sin 2\varphi}{2} \right|_0^{2\pi} + \frac{\rho a^2 h^3 \pi}{12}$$

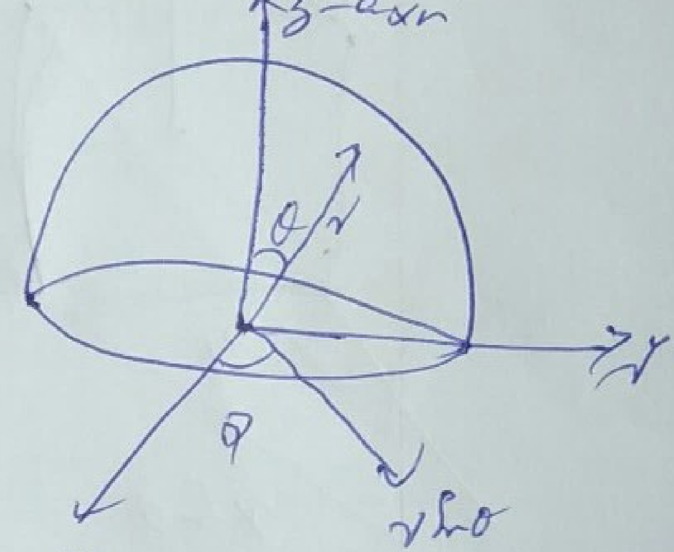
$$= \rho \frac{h a^4}{8}(2\pi) + \frac{\rho a^2 h^3 \pi}{12}$$

$$= \frac{\rho \pi h a^4}{4} + \frac{\rho a^2 \pi h^3}{12} = \frac{\rho \pi a^2 h}{12}(3a^2 + h$$

using value of ρ, we get

$= \frac{1}{12} M(3a^2 + h^2)$ is required moment of inertia about y-axis. Note that it is same as m.o.i about x-axis

Moment of inertia of a hemisphere about its axis of symmetry.



Let axis of symmetry be z-axis, then we have to find I_{zz}

Since
$$I_{zz} = \sum_{i=1}^{n} (x_i^2 + y_i^2) m_i$$

$$\Rightarrow \quad I_{zz} = \int_V (x^2 + y^2) \rho \, dV \quad (1)$$

In case of a sphere (or hemisphere) we have

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

So
$$x^2 + y^2 = r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi$$
$$= r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) = r^2 \sin^2\theta$$

Also $dV = (dr)(r\,d\theta)(r \sin\theta \, d\phi)$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and limits for hemisphere are

$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi, \quad 0 \le r \le a$$

Hence from ①

$$I_{zz} = \rho \int_0^{\varphi}\int_0^{\pi/2}\int_0^{a} r^2 \sin^2\theta \, r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \rho \iiint r^4 \sin^3\theta \, dr\, d\theta\, d\varphi$$

$$= \rho \iint \frac{r^5}{5}\Big|_0^a \sin^3\theta \, d\theta\, d\varphi$$

$$= \rho \iint \frac{a^5}{5} \sin^3\theta \, d\theta\, d\varphi$$

$$= \rho \frac{a^5}{5} \int_0^{\pi/2} |\varphi|_0^{2\pi} \sin^3\theta \, d\theta = \frac{2}{3} M a^2$$

by $\rho = \frac{M}{V}$

x ———— x

ii about an axis through the base and perp. to the axis of symmetry

i.e. we have to find I_{xx} or I

$$I_{xx} = \Sigma m_i (y_i^2 + z_i^2)$$

$$\Rightarrow I_{xx} = \int_V (y^2 + z^2) \rho \, dV$$

$$= \iiint (r^2 \sin^2\theta \sin^2\varphi + r^2 \cos^2\theta) \rho r^2 \sin\theta$$

$$= \rho \iiint r^4 (\sin^2\theta \sin^2\varphi + \cos^2\theta) \sin\theta \, dr\, d\theta$$

$$= \rho \iiint r^4 (\sin^3\theta \sin^2\varphi + \cos^2\theta \sin\theta) dr\, d\theta$$

— do.

To find moment of inertia for a uniform sphere of radius 'a' about axis through its centre

— do —

Note: For sphere, $I_{xx} = I_{yy} = I_{zz}$ (do)

use $x = r\sin\theta\cos\phi$ $\quad 0 \le r \le a$
$y = r\sin\theta\sin\phi$ $\quad 0 \le \theta \le \pi$
$z = r\cos\theta$ $\quad 0 \le \phi \le 2\pi$

and $\Delta V = \Delta r(r\Delta\theta)(r\sin\theta\Delta\phi)$

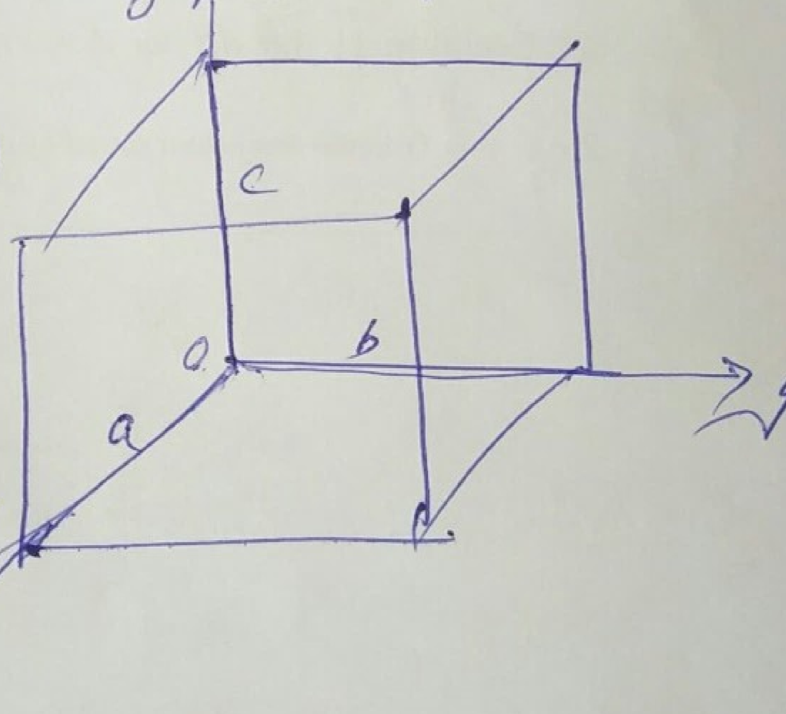
To find moment of inertia of a uniform solid rectangular box at one of its corners

Here

$I_{xx} = \Sigma m_i(y_i^2+z_i^2)$

$\Rightarrow I_{xx} = \int_V (y^2+z^2)\rho\, dV$

$= \int_0^c\int_0^b\int_0^a (y^2+z^2)\rho\, dx\, dy\, dz$



Since for a small box

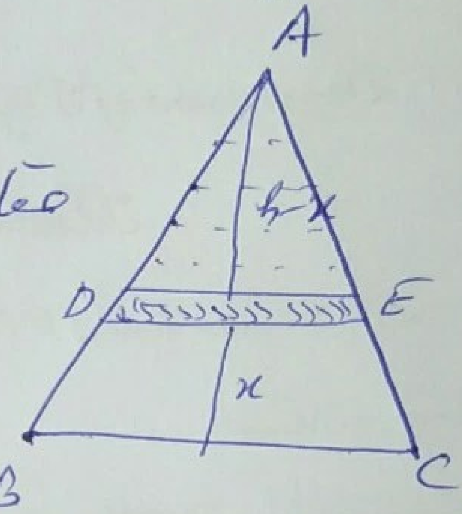
$\Delta V = \Delta x\,\Delta y\,\Delta z$

$\boxed{\rho = \frac{M}{abc}}$

where as $0 \le x \le a$, $0 \le y \le b$

and $0 \le z \le c$ — do —

To find moment of inertia of a triangular lamina about its one side



To find moment of inertia about the side BC of the lamina, we divide the lamina into rectangular strips taken parallel to BC. If x is $\perp$ distance of the ith strip from BC, then

$$I = \sum x_i^2 m_i$$

$$= \int x^2\,dm = \int x^2 \rho\, dA \quad (1)$$

From similar Δs ADE and ABC

$$\frac{|DE|}{|BC|} = \frac{h-x}{h}$$

$$\frac{|DE|}{a} = \frac{h-x}{h} \Rightarrow |DE| = \left(\frac{h-x}{h}\right)a$$

If Δx is width of the strip then $\Delta m = \rho \Delta A = \rho\left(\frac{h-x}{h}\right) a\,\Delta x$

Hence from (1)

$$I = \int_0^h x^2 \rho a \left(\frac{h-x}{h}\right) dx$$

$$= \frac{\rho a}{h}\int_0^h (hx^2 - x^3)\,dx$$

$$= \frac{\rho a}{h}\left| h\frac{x^3}{3} - \frac{x^4}{4}\right|_0^h$$

$$= \frac{\rho a}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right)$$

$$= \rho \frac{a}{h} \frac{h^4}{12} = \frac{\rho a h^3}{12} = \left(\frac{M}{\frac{1}{2} a h}\right) \frac{ah}{12}$$

$$= \frac{1}{6} M h^2 \quad \text{Ans}$$

To find i moment of inertia for an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

ii find moment of inertia of a solid cone about its axis and about an axis through its base and $\perp$ to the axis of the cone.