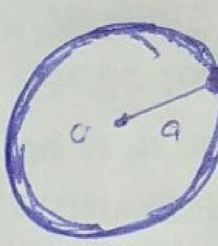
Moment of inertia of a uniform ring about an axis through its centre.

Let M be the mass and a be the radius of the ring. Then the ~~circ~~ circumference of the ring is $2\pi a$.



Moment of inertia about an axis through its centre is

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} a^2 \Delta m_i \quad (1)$$

Now mass density relation gives

$\Delta m_i = \rho \Delta l_i$, Δl is length of an element of ring

$\Rightarrow \Delta m_i = \rho \Delta s_i$

Hence from (1)

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} a^2 \rho \Delta s_i$$

$$= \int_0^{2\pi a} \rho a^2 ds, \text{ where } 2\pi a \text{ is entire length of the ring.}$$

$$= \rho a^2 \left| s \right|_0^{2\pi a}$$

$$= \rho a^2 (2\pi a)$$

$$= \rho \, 2\pi a^3$$

$$= \frac{M}{2\pi a}(2\pi a^3)$$

$$= Ma^2$$

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Moment of inertia of a uniform circular disc.

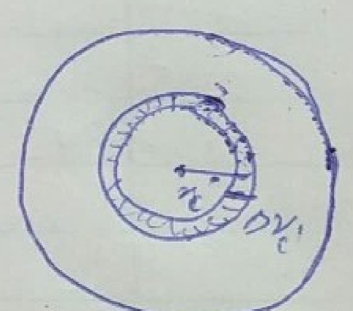
Let M and a are the mass and radius of the disc respectively.

A circular disc can be assumed as a collection of circular strips of radii r_i and width Δr_i.

As circumference of any ith strip is $2\pi r_i$, so that area of the ith strip will be $2\pi r_i \Delta r_i$ and

hence mass of the strip $= \Delta m_i = \rho\, 2\pi r_i \Delta r_i$

The moment of inertia of the ~~strip~~ disc is given by



$$I = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta m_i r_i^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} r_i^2 (\rho 2\pi r_i \Delta r_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\rho\, r_i^3 \Delta r_i$$

$$= \int_0^a 2\pi\rho\, r^3 dr$$

$$= 2\pi\rho \left| \frac{r^4}{4} \right|_0^a$$

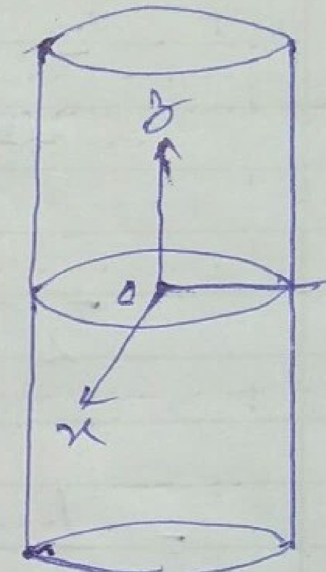
$$= 2\pi\rho \frac{a^4}{4}$$

$$= 2\pi \frac{M}{\pi a^2} \frac{a^4}{4} \quad \because \pi a^2 \text{ is area of the disc}$$

$$= \frac{1}{2} M a^2 \quad \text{Ans.}$$

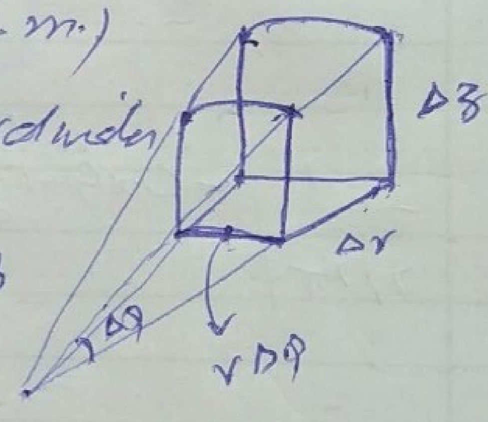
Moment of inertia of a uniform circular cylinder about i) its axis and ii) an axis ⊥ through its middle and ⊥ to the axis of the cylinder.

i) About axis of cylinder.



Let a is radius and h height of the cylinder. Axis of the cylinder be taken along z-axis, so that we have to find I_{zz}. Let O be taken as origin at the mid of the cylinder (c.m.)

Since cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$



So moment of inertia is given by

$$I_{zz} = \int_V (x^2+y^2)\,\rho\, dV \quad (1)$$

where dV will be obtained from the volume element of the cylinder and is given by $\Delta V = \Delta r \times r\Delta\theta \times \Delta z$

$$= r\,\Delta r\,\Delta\theta\,\Delta z$$

Also for cylindrical coordinates we have $0 \le r \le a$, $0 \le \theta \le 2\pi$, $-\frac{h}{2} \le z \le \frac{h}{2}$

so that from (1)

$$I_{zz} = \int_0^a\int_0^{2\pi}\int_{-h/2}^{h/2} (r^2\cos^2\theta + r^2\sin^2\theta)\,\rho\, r\, dr\, d\theta\, dz$$

$$= \rho \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a r^3 \, dr \, d\phi \, dz$$

$$= \rho \int_{-h/2}^{h/2}\int_0^{2\pi} \left(\frac{r^4}{4}\right|_0^a d\phi \, dz$$

$$= \rho \int_{-h/2}^{h/2}\int_0^{2\pi} \frac{a^4}{4} \, d\phi \, dz$$

$$= \rho \frac{a^4}{4} \int_{-h/2}^{h/2} |\phi|_0^{2\pi} \, dz = \rho \frac{a^4}{4}(2\pi) \int_{-h/2}^{h/2} 1 \, dz$$

$$= \frac{\pi \rho a^4}{2} |z|_{-h/2}^{h/2} = \frac{\pi \rho a^4}{2}\left(\frac{h}{2}+\frac{h}{2}\right)$$

$$= \frac{\pi \rho a^4}{2}(h) = \frac{\pi h a^4}{2}\rho = \frac{\pi h a^4}{2}\left(\frac{M}{\pi a^2 h}\right)$$

$$= \frac{1}{2} M a^2 \quad \text{Ans.}$$

ii) About an axis perp. to the axis of the cylinder and through its centre.

Here we have to find I_{xx} or I_{yy}

$$I_{xx} = \int_v (y^2 + z^2) \rho \, dv$$

$$\Rightarrow \quad I_{xx} = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a (r^2 \sin^2\phi + z^2) \rho \, r \, dr \, d\phi \, dz$$

$$= \iiint \rho (r^3 \sin^2\phi + z^2 r) \, dr \, d\phi \, dz$$

$$= \iint \rho \left(\frac{r^4}{4} \sin^2\phi + z^2 \frac{r^2}{2} \right|_0^a d\phi \, dz$$

$$= \iint \rho \left(\frac{a^4}{4} \sin^2\phi + \frac{a^2}{2} z^2 \right) d\phi \, dz$$

$$= \int \rho \left(\frac{a^4}{4} \sin^2\phi \, z + \frac{a^2}{2} \frac{z^3}{3} \right|_{-h/2}^{h/2} d\phi$$

$$= \int_0^{2\pi} \rho \left(\frac{a^4}{4} \sin^2\theta \, (h) + \frac{a^2}{2} \, \frac{2h^3}{3\times 8} \right)$$

$$= \rho \int_0^{2\pi} \left(\frac{h a^4}{4} \sin^2\theta + \frac{a^2 h^3}{24} \right) d\theta$$

$$= \rho \left[\int_0^{2\pi} \frac{h a^4}{8} (2 \sin^2\theta) \, d\theta + \frac{a^2 h^3}{24} \left| \theta \right|_0^{2\pi} \right]$$

$$= \rho \left[\int_0^{2\pi} \frac{h a^4}{8} (1 - \cos 2\theta) \, d\theta + \frac{a^2 h^3}{24} (2\pi) \right]$$

$$= \rho \left[\frac{h a^4}{8} \left| \theta - \frac{\sin 2\theta}{2} \right|_0^{2\pi} + \frac{a^2 h^3 \pi}{12} \right]$$

$$= \rho \left[\frac{h a^4}{8} (2\pi) + \frac{a^2 h^3 \pi}{12} \right]$$

$$= \rho \left[\frac{a^4 h \pi}{4} + \frac{a^2 h^3 \pi}{12} \right]$$

$$= \rho \left(\frac{a^2}{1} + \frac{h^2}{3} \right) \frac{a^2 h \pi}{4}$$

$$= \rho \, (3a^2 + h^2) \frac{\pi a^2 h}{12}$$

$$= \frac{M}{\pi a^2 h} (3a^2 + h^2) \frac{\pi a^2 h}{12}$$

$$= \frac{1}{12} (3a^2 + h^2) M.$$

is moment of inertia

[illegible]