Now we shall discuss some physical systems for finding moment of inertia. The following points may be noted before going into details.

i. In case of a uniform rod, we shall assume the rod is composed of infinite numbers of small element of the rod such that mass of the ith element would be $\Delta m_i = \rho \Delta x_i$, where Δx is length (only) of the element.

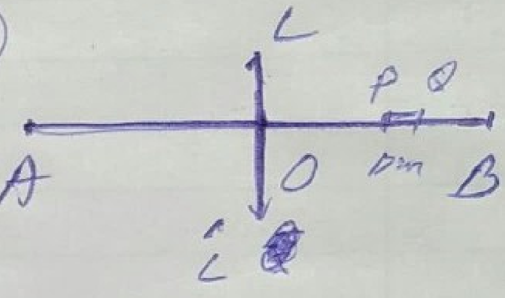
ii. In case of a plane lamina, we shall assume that the lamina is composed of infinite many small rectangular element, so that the mass of one of the element would be taken $\Delta m_i = \rho \Delta A_i = \rho \Delta x_i \Delta y_i$, where $\Delta x, \Delta y$ are dimension of the element.

iii. In case of a rigid body, a rigid body is a collection of infinite many small rectangular parallelopiped shape elements.

the mass of the ith infinitesimal element is $\Delta m_i = \rho \Delta V_i = \rho \Delta x_i \Delta y_i \Delta z_i$,
where $\Delta V = \Delta x_i \Delta y_i \Delta z_i$ is volume of the element.

§ Moment of inertia of a uniform rod, about ~~an~~ axis through its middle.

Let $AB = 2a$ is length of the uniform rod of density ρ.



We want to find moment of inertia about a $\perp$ axis LOL' through its middle O (origin)

Consider an element PQ of mass Δm_i at a distance x_i from O

Let Δx_i is width of the element

Then mass of the element $= \Delta m_i = \rho \Delta x_i$

For moment of inertia, we have

$$I_{LL'} = \lim_{n \to \infty} \sum_{i=1}^{n} m_i d_i^2$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta m_i x_i^2$$

$$= \lim_{n\to\infty} \sum_{n=1}^{\infty} x_i^2 \rho \Delta x_i$$

$$= \int_{-a}^{a} \rho x^2 dx$$

$$= \rho \left|\frac{x^3}{3}\right|_{-a}^{a} = \rho\left(\frac{a^3}{3} + \frac{a^3}{3}\right)$$

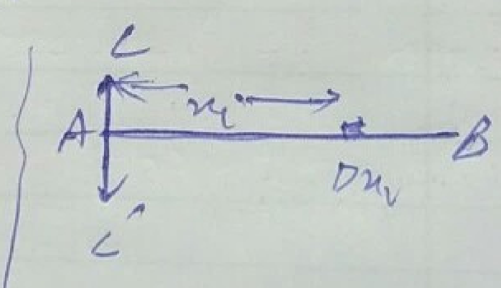
$$= 2\rho\frac{a^3}{3}$$

$$= 2\frac{M}{2a}\frac{a^3}{3} \qquad \because \rho = \frac{M}{2a}$$

$$= \frac{1}{3}Ma^2 \quad \text{Ans}$$

§ Moment of inertia of a uniform rod about a $\perp$ axis passing through one end point of the rod.

Consider a $\perp$ axis LL' through end point A of the rod $AB = 2a$.



Assuming A as origin and LL' as y-axis, we have moment of inertia ~~the~~ about LL'.

$$I_{yy} = \lim_{n\to\infty} \sum_{i=1}^{n} m_i d_i^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta m_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \rho \Delta x_i$$

$$I_{yy} = \int_0^{2a} x^2 \rho\, dx$$

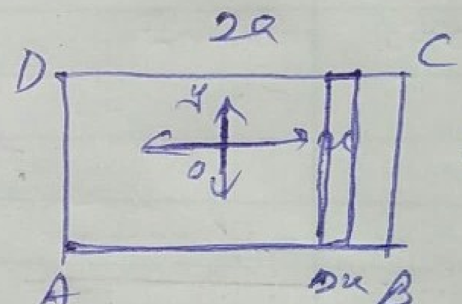
$$= \rho \left| \frac{x^3}{3} \right|_0^{2a} = \rho \frac{8a^3}{3}$$

$$= \frac{M}{2a} \times \frac{8a^3}{3} \quad \text{by using } \rho = \frac{M}{2a}$$

$$= \frac{4Ma^2}{3}$$

Moment of inertia of uniform rectangular lamina (planer body).

Let ABCD is a rectangular lamina of dimension $2a \times 2b$.



Let O be origin at the middle of the lamina and sides of lamina are taken parallel to x-axis and y-axis.

Firstly, we divide the lamina by taking rectangular strips (parallel to y-axis) of width Δx_i and length $2b$. Then the mass of the ith strip would be $\Delta m_i = \rho\, \Delta A_i =$

$$= \rho\, \Delta x_i \times 2b$$

$$= 2b \rho\, \Delta x_i$$

So moment of inertia about y-axis can be found as following

$$I_{yy} = \lim_{n \to \infty} \sum_{i=1}^{n} m_i d_i^2$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} x_i^2 \Delta m_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} x_i^2 \rho (2b) \Delta x_i$$

$$= \int_{-a}^{a} x^2 \rho (2b)\, dx$$

$$= 2b\rho \int_{-a}^{a} x^2\, dx$$

$$= 2b\rho \left| \frac{x^3}{3} \right|_{-a}^{a}$$

$$= 2b\rho \left(\frac{2a^3}{3} \right)$$

$$= \rho \frac{4a^3 b}{3}$$

$$= \frac{M}{2a \times 2b} \times \frac{4a^3 b}{3}$$

$$I_{yy} = \frac{1}{3} M a^2$$

Similarly, we can find

$$I_{xx} = \frac{1}{3} M b^2 \text{ do.}$$