Momental Ellipsoid or Ellipsoid of inertia

since moment of inertia of a system of particles about any line l is

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz} + 2\lambda\mu I_{xy} + 2\lambda\nu I_{xz} + 2\mu\nu I_{yz}$$

If we use $\lambda = \cos\alpha = \frac{x}{r}$

$\mu = \cos\beta = \frac{y}{r}$ and $\nu = \cos\gamma = \frac{z}{r}$

into (1), we get

$$I_l = \frac{x^2}{r^2} I_{xx} + \frac{y^2}{r^2} I_{yy} + \frac{z^2}{r^2} I_{zz} + 2\frac{x}{r}\frac{y}{r} I_{xy}$$
$$+ 2\frac{x}{r}\frac{z}{r} I_{xz} + 2\frac{y}{r}\frac{z}{r} I_{yz}$$

which gives

$$I_{xx} x^2 + I_{yy} y^2 + I_{zz} z^2 + 2 I_{xy} xy + 2 I_{xz} xz$$
$$+ 2 I_{yz} yz = r^2 I_l$$

$$I_{xx} x^2 + I_{yy} y^2 + I_{zz} z^2 + 2 I_{xy} xy + 2 I_{xz} xz + 2 I_{yz}$$

where $r^2 I_l = c$ (Sa

This eq. (2) represents equation of an ellipsoid, known as momental ellipsoid or ellipsoid of moment of inertia

§ Perpendicular axes Theorem :-

Moment of inertia of a plane rigid body about an axis perpendicular to the body is equal to the sum of the moments of inertias about two mutually perpendicular axes lying in the plane of the body and intersecting at a common point with the given axis.

If we choose plane rigid by as xy-plane and z-axis as perpendicular to the rigid body, then the above statement implies

$$I_{zz} = I_{xx} + I_{yy}$$

$$\text{or} \quad I_{33} = I_{11} + I_{22}$$

where I_{11} or I_{xx}, I_{22} or I_{yy}, I_{33} or I_{zz} are moment of inertias about x-axis, y-axis and z-axis respectively.

Proof Choosing z-axis as a perpendicular axis to the laminar rigid body, we have

$$I_{zz} = \sum_{i=1}^{n} m_i d_i^2$$

where d_i is perpendicular distance of the ith particle, in the plane, from the z-axis. In xy-plane, we have

$$d_i^2 = x_i^2 + y_i^2$$

So $$I_{zz} = \Sigma m_i (x_i^2 + y_i^2)$$

$$= \Sigma m_i x_i^2 + \Sigma m_i y_i^2$$

$$= I_{yy} + I_{xx}$$

$$I_{zz} = I_{xx} + I_{yy}$$ prove

since x_i = perpendicular from the ith particle to y-axis, and

y_i = perdicular from the ith particle to x-axis.

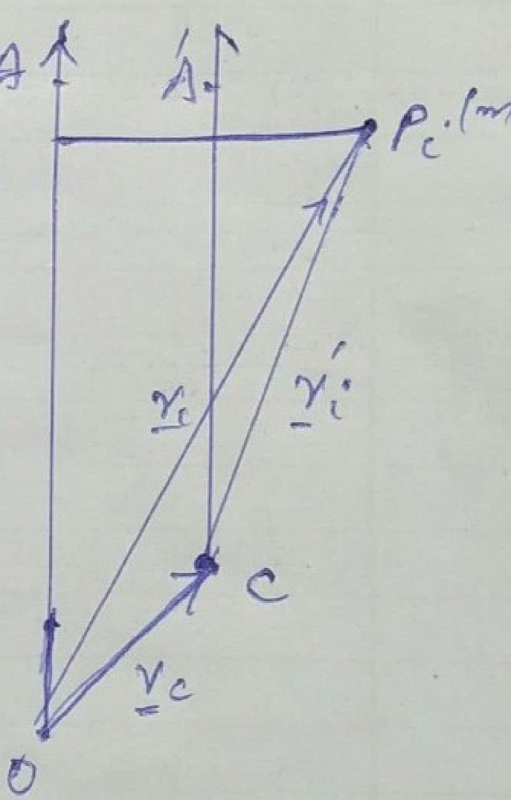
x———x

Parallel axes Theorem :-

The moment of inertia of a rigid body about an axis is equal to the sum of moment of inertia about a parallel axis through the centroid and the moment of inertia due to total mass placed at the centroid (moment of inertia of the centroid)

Proof

If I denotes moment of inertia about a given axis $\vec{OA}$ and d_i is $\perp$ distance of the ith particle from the axis $\vec{OA}$, then



by def.

$$I = \Sigma m_i d_i^2$$

$$= \Sigma m_i |\hat{e} \times \underline{r}_i| \quad ①$$

where $\hat{e}$ is unit vector in the direction of the axis $\vec{OA}$.

Let C is centroid of the rigid body with $\underline{r}_c$ as position vector, draw an axis $\vec{CA'}$ parallel to $\vec{OA}$.

Let $\underline{r}_i'$ is position vector of P_i with respect to C, then

$$\underline{r}_i = \underline{r}_c + \underline{r}_i' \quad (2)$$

By (1) and (2), we have

$$I = \Sigma m_i (\hat{e} \times (\underline{r}_c + \underline{r}_i'))^2$$

$$= \Sigma m_i (\hat{e} \times \underline{r}_c + \hat{e} \times \underline{r}_i')^2$$

$$= \Sigma m_i ((e \times \underline{r}_c)^2 + (e \times r_i')^2 + 2(e \times \underline{r}_c)\cdot(e \times r_i'))$$

$$= \Sigma m_i (e \times \underline{r}_c)^2 + \Sigma m_i (e \times r_i')^2 + 2(e \times \underline{r}_c)\cdot \Sigma m_i (e \times \underline{r}_i')$$

$$= \Sigma m_i d_c^2 + \Sigma m_i d_i'^2 + 2(\hat{e} \times \underline{r}_c)\cdot(\hat{e} \times \Sigma m_i \underline{r}_i')$$

$$= \Sigma m_i d_c^2 + \Sigma m_i d_i'^2 + 0$$

$\because \Sigma m_i \underline{r}_i' = 0$ for c.m.

$$= (\Sigma m_i) d_c^2 + \Sigma m_i d_i'^2$$

$$= M d_c^2 + I'$$

$$= I_c + I'$$

where I_c is moment of inertia of the centroid (whole mass) and I' is moment of inertia of the system of particles about new parallel axis