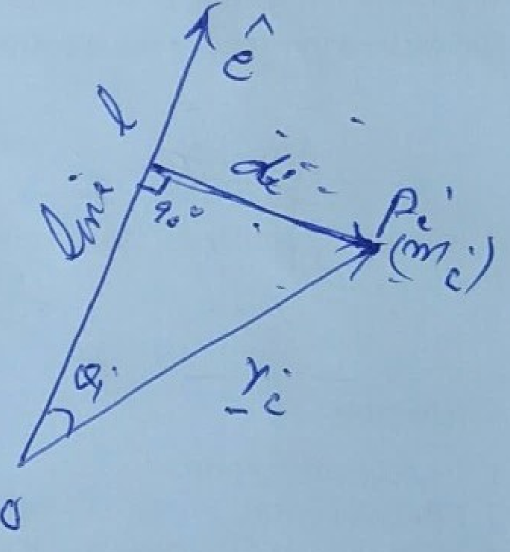
Moment of inertia of a rigid body about a given line.

Let $M = \Sigma m_i$ is mass of a rigid body where $m_1, m_2, \cdots m_n$ is collection of n masses of the system. ~~Since~~ Now moment of inertia for a system of n particles, about any line l, is

$$I_l = \Sigma m_i d_i^2 \quad (1)$$

where d_i is perpendicular distance of the mass m_i from the line l.

Now $\frac{d_i}{|OP_i|} = \sin\theta_i$ gives



$$d_i = |OP_i| \sin\theta_i$$

or $d_i = |\underline{r}_i| \sin\theta_i \quad (II)$

Also $|\hat{e} \times \underline{r}_i| = |e||\underline{r}_i| \sin\theta_i$

$\Rightarrow |\hat{e} \times \underline{r}_i| = r_i \sin\theta_i$, where $\hat{e}$ is unit vector along the line.

So that (2) becomes

$$d_i = |e \times \underline{r}_i|$$

and hence from (1)

$$I_l = \sum_{i=1}^{n} m_i |\hat{e} \times \underline{r}_i|^2 \quad (3)$$

If $\hat{e} = \lambda i + \mu j + \nu k$, where λ, μ, ν are direction cosines of the line l and

$\underline{r}_i = x_i i + y_i j + z_i k$, then

$$\hat{e} \times \underline{r}_i = \begin{vmatrix} i & j & k \\ \lambda & \mu & \nu \\ x_i & y_i & z_i \end{vmatrix}$$

$$= (\mu z_i - \nu y_i) i + (\nu x_i - \lambda z_i) j + (\lambda y_i - \mu x_i) k$$

$$\Rightarrow |\underline{e} \times \underline{r}_i|^2 = (\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2$$

$$= \mu^2 z_i^2 + \nu^2 y_i^2 - 2\mu\nu y_i z_i + \nu^2 x_i^2 + \lambda^2 z_i^2$$
$$- 2\nu\lambda x_i z_i + \lambda^2 y_i^2 + \mu^2 x_i^2 - 2\lambda\mu x_i y_i$$

$$= \lambda^2 (y_i^2 + z_i^2) + \mu^2 (x_i^2 + z_i^2) + \nu^2 (x_i^2 + y_i^2)$$
$$- 2\mu\lambda x_i y_i - 2\lambda\nu x_i z_i - 2\mu\nu y_i z_i$$

using this value of $|e \times r_i|^2$ into eq ③ we get

$$I_l = \Sigma m_i \{ \lambda^2 (y_i^2 + z_i^2) + \mu^2 (x_i^2 + z_i^2) + \nu^2 (x_i^2 + y_i^2)$$
$$- 2\mu\lambda x_i y_i - 2\lambda\nu x_i z_i - 2\mu\nu y_i z_i \}$$

$$= \lambda^2 \Sigma m_i (y_i^2 + z_i^2) + \mu^2 \Sigma m_i (x_i^2 + z_i^2)$$
$$+ \nu^2 \Sigma m_i (x_i^2 + y_i^2) - 2\lambda\mu \Sigma m_i x_i y_i$$
$$- 2\lambda\nu \Sigma m_i x_i z_i - 2\mu\nu \Sigma m_i y_i z_i$$

Since $I_{xx} = \Sigma m_i (y_i^2 + z_i^2)$ & $I_{xy} = -\Sigma m_i x_i y_i$

$I_{yy} = \Sigma m_i (x_i^2 + z_i^2)$ & $I_{yz} = -\Sigma m_i y_i z_i$

$I_{zz} = \Sigma m_i (x_i^2 + y_i^2)$ & $I_{zx} = -\Sigma m_i z_i x_i$

So that

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + \nu^2 I_{zz}$$
$$+ 2\lambda\mu I_{xy} + 2\lambda\nu I_{xz} + 2\mu\nu I_{yz}$$

The above relation can also be written as

$$I_l = \lambda^2 A + \mu^2 B + \nu^2 C + 2\mu\nu D + 2\lambda\nu E + 2\lambda\mu F \longrightarrow (4)$$

where $A = I_{xx} = I_{11}$, $B = I_{yy} = I_{22}$, $C = I_{33} = I_{zz}$

$D = I_{yz} = I_{23}$, $E = I_{xz} = I_{13}$, $F = I_{12} = I_{xy}$

The relation (4) gives the moment of inertia of a system of particles m_i, r_i about any line l, with direction cosines λ, μ, ν or unit vector $e = \lambda\hat{i} + \mu\hat{j} + \nu\hat{k}$

x~~~~~~~~x

Momental Ellipsoid or Ellipsoid of Inertia