Pt I

§ Angular Momentum of a particle:—

Moment of momentum of any particle is called angular momentum. If m is the mass of the body and v its velocity, then its momentum is given by $\underline{p} = m\underline{v}$

Moment of this momentum is

$$\underline{L} = \underline{r} \times \underline{p}$$

where r is position vector (radius vector) of the body and $\underline{L}$ denotes the angular momentum. Further since $\underline{v} = \underline{\omega} \times \underline{r}$

so the relation of angular momentum becomes

$$\underline{L} = \underline{r} \times m\underline{v} = m\, \underline{r} \times \underline{v}$$

$$= m\, \underline{r} \times (\underline{\omega} \times \underline{r})$$

$$= m\big((\underline{r}\cdot\underline{r})\,\omega - (\underline{r}\cdot\underline{\omega})\,\underline{r}\big)$$

$$= m\big(r^2\underline{\omega} - \underline{r}\cdot\underline{\omega}\,\underline{r}\big)$$

Another notation for the A.M. is "J".

a — a

§ Angular Momentum of a system of particles.

Consider a system of n particle having masses $m_1, m_2, \cdots m_n$ such that their respective radius vectors are given by $\underline{r}_1, \underline{r}_2, \cdots \underline{r}_n$. ~~For any rth particle are~~

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If v_i is linear velocity of any ith particle, Then we can write

$$\underline{v}_i = \underline{\omega} \times \underline{r}_i \text{ , where } \underline{r}_i \text{ is}$$

the position vector of the ith particle.

It is also assumed that each particle is moving with same angular velocity $\underline{\omega}$.

So the angular ~~velocity~~ momentum of the ith particle is

$$L_i = \underline{r}_i \times \underline{P}_i$$

or $$L_i = \underline{r}_i \times m_i v_i = m_i \underline{r}_i \times v_i$$

$$L_i = m_i \, r_i \times (\omega \times r_i)$$

So for the entire system, the total angular momentum becomes,

$$L = \sum_{i=1}^{n} L_i \qquad \boxed{L = J}$$

$$= \Sigma m_i \, \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$$

$$= \Sigma m_i ((\underline{r}_i \cdot \underline{r}_i) \underline{\omega} - (\underline{r}_i \cdot \underline{\omega}) \underline{r}_i) \quad (1)$$

using $$\underline{r}_i = x_i i + y_i j + z_i k$$

$$\underline{\omega} = \omega_x i + \omega_y j + \omega_z k$$

and $$\underline{L} = L_x i + L_y j + L_z k$$

we get from (1)

$$L_x \hat{i} + L_y \hat{j} + L_z \hat{k} = \Sigma m_i \{ r_i^2 \underline{\omega} - (\underline{r_i} \cdot \underline{\omega}) \underline{r_i} \}$$

$$= \Sigma m_i \{ (x_i^2 + y_i^2 + z_i^2)(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$$

$$- (x_i \omega_x + y_i \omega_y + z_i \omega_z)(x_i \hat{i} + y_i \hat{j} + z_i \hat{k})$$

Now taking "$\hat{x}$" component on both sides

$$L_x = \Sigma m_i \{ (x_i^2 + y_i^2 + z_i^2)\omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \}$$

$$= \Sigma m_i \{ (x_i^2 + y_i^2 + z_i^2)\omega_x - (x_i^2 \omega_x + \omega_y x_i y_i + \omega_z x_i z_i) \}$$

$$= \Sigma m_i \{ (y_i^2 + z_i^2)\omega_x - (x_i y_i \omega_y + x_i z_i \omega_z) \}$$

$$= \Sigma m_i (y_i^2 + z_i^2)\omega_x - \Sigma m_i \omega_y x_i y_i - \Sigma m_i \omega_z x_i z_i$$

$$= \left(\Sigma m_i (y_i^2 + z_i^2) \right) \omega_x + \left(-\Sigma m_i x_i y_i \right) \omega_y + \left(-\Sigma m_i x_i z_i \right) \omega_z$$

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

where $I_{xx} = \Sigma m_i (y_i^2 + z_i^2)$

$$I_{xy} = -\Sigma m_i x_i y_i$$

$$I_{xz} = -\Sigma m_i x_i z_i$$

are known as moment of inertia and product of inertia along x-axis.

Similarly for components along y-axis and z-axis we get $L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$$I_{yy} = \Sigma m_i(z_i^2 + x_i^2), \quad I_{yz} = -\Sigma m_i z_i y_i$$

and $L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$ and $I_{yx} = -\Sigma m_i y_i x_i$

have $I_{zz} = \Sigma m_i(x_i^2 + y_i^2)$

$$I_{zx} = -\Sigma m_i z_i x_i \quad \text{and} \quad I_{zy} = -\Sigma m_i z_i y_i$$

The three moment of inertia can also be written as

$$I_{xx} = \Sigma m_i(r_i^2 - x_i^2)$$

$$I_{yy} = \Sigma m_i(r_i^2 - y_i^2)$$

$$I_{zz} = \Sigma m_i(r_i^2 - z_i^2)$$

The three eqs of angular momentum

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

can also be written as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

where as $\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ \ldots \end{pmatrix}$ is called inertia mat

Moment of inertia for a rigid body :-

We have calculated M. of I for a system of n particles e.g. $I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$ (1) etc

A rigid body can be considered as a collection of infinite many infinitesimal particles with no gap b/w them.

So considering the mass of the infinitesimal ith particle $\Delta m_i = \rho(r_i) \Delta V_i$, where ΔV_i is volume of ~~the~~ and $\rho(r)$ is the density of the ith particle, we have from (1)

$$I_{xx} = \lim_{n \to \infty} \sum_{i=1}^{n} (y_i^2 + z_i^2) \Delta m_i$$

$$= \lim_{n \to \infty} \sum (y_i^2 + z_i^2) \rho(r_i) \Delta V_i$$

$$= \int_V \rho(r) (y^2 + z^2) \, dV$$

Similarly $I_{yy} = \int_V \rho(r) (z^2 + x^2) \, dV$

$$I_{zz} = \int_V \rho(r) (x^2 + y^2) \, dV$$

and the corresponding product of inertia are

$$I_{xy} = -\int \rho(r) \, xy \, dV$$

$$I_{yz} = -\int \rho(r) \, yz \, dV$$

and $I_{zx} = -\int \rho(r) \, zx \, dV$