Theorem 18 *There are disjoint sets of real numbers A and B for which*

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof We prove this by contradiction. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then, by the very definition of measurable set, every set must be measurable. This contradicts the preceding theorem. □

PROBLEMS

29. (i) Show that rational equivalence defines an equivalence relation on any set.
    (ii) Explicitly find a choice set for the rational equivalence relation on $\mathbf{Q}$.
    (iii) Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbf{R}$? Is this an equivalence relation on $\mathbf{Q}$?
30. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.
31. Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.
32. Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?
33. Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which
$$m^*(E) = m^*(G), \text{ while } m^*(G \sim E) > 0.$$

2.7 THE CANTOR SET AND THE CANTOR-LEBESGUE FUNCTION

We have shown that a countable set has measure zero and a Borel set is Lebesgue measurable. These two assertions prompt the following two questions.

Question 1 If a set has measure zero, is it also countable?

Question 2 If a set is measurable, is it also Borel?

The answer to each of these questions is negative. In this section we construct a set called the Cantor set and a function called the Cantor-Lebesgue function. By studying these we answer the above two questions and later provide answers to other questions regarding finer properties of functions.

Consider the closed, bounded interval $I = [0, 1]$. The first step in the construction of the Cantor set is to subdivide I into three intervals of equal length $1/3$ and remove the interior of the middle interval, that is, we remove the interval $(1/3, 2/3)$ from the interval $[0, 1]$ to obtain the closed set C_1, which is the union of two disjoint closed intervals, each of length $1/3$:

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

We now repeat this "open middle one-third removal" on each of the two intervals in C_1 to obtain a closed set C_2, which is the union of 2^2 closed intervals, each of length $1/3^2$:

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

We now repeat this "open middle one-third removal" on each of the four intervals in C_2 to obtain a closed set C_3, which is the union of 2^3 closed intervals, each of length $1/3^3$. We continue this removal operation countably many times to obtain the countable collection of sets $\{C_k\}_{k=1}^{\infty}$. We define the Cantor set $\mathbf{C}$ by

$$\mathbf{C} = \bigcap_{k=1}^{\infty} C_k.$$

The collection $\{C_k\}_{k=1}^{\infty}$ possesses the following two properties:

(i) $\{C_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets;

(ii) For each k, C_k is the disjoint union of 2^k closed intervals, each of length $1/3^k$.

Proposition 19 *The Cantor set* $\mathbf{C}$ *is a closed, uncountable set of measure zero.*

Proof The intersection of any collection of closed sets is closed. Therefore $\mathbf{C}$ is closed. Each closed set is measurable so that each C_k and $\mathbf{C}$ itself is measurable.

Now each C_k is the disjoint union of 2^k intervals, each of length $1/3^k$, so that by the finite additivity of Lebesgue measure,

$$m(C_k) = (2/3)^k.$$

By the monotonicity of measure, since $m(\mathbf{C}) \leq m(C_k) = (2/3)^k$, for all k, $m(\mathbf{C}) = 0$. It remains to show that $\mathbf{C}$ is uncountable. To do so we argue by contradiction. Suppose $\mathbf{C}$ is countable. Let $\{c_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbf{C}$. One of the two disjoint Cantor intervals whose union is C_1 fails to contain the point c_1; denote it by F_1. One of the two disjoint Cantor intervals in C_2 whose union is F_1 fails to contain the point c_2; denote it by F_2. Continuing in this way, we construct a countable collection of sets $\{F_k\}_{k=1}^{\infty}$, which, for each k, possesses the following three properties: (i) F_k is closed and $F_{k+1} \subseteq F_k$; (ii) $F_k \subseteq C_k$; and (iii) $c_k \notin F_k$. From (i) and the Nested Set Theorem[12] we conclude that the intersection $\bigcap_{k=1}^{\infty} F_k$ is nonempty. Let the point x belong to this intersection. By property (ii),

$$\bigcap_{k=1}^{\infty} F_k \subseteq \bigcap_{k=1}^{\infty} C_k = \mathbf{C},$$

and therefore the point x belongs to $\mathbf{C}$. However, $\{c_k\}_{k=1}^{\infty}$ is an enumeration of $\mathbf{C}$ so that $x = c_n$ for some index n. Thus $c_n = x \in \bigcap_{k=1}^{\infty} F_k \subseteq F_n$. This contradicts property (iii). Hence $\mathbf{C}$ must be uncountable. □

A real-valued function f that is defined on a set of real numbers is said to be **increasing** provided $f(u) \leq f(v)$ whenever $u \leq v$ and said to be **strictly increasing**, provided $f(u) < f(v)$ whenever $u < v$.

We now define the Cantor-Lebesgue function, a continuous, increasing function φ defined on $[0, 1]$ which has the remarkable property that, despite the fact that $\varphi(1) > \varphi(0)$, its derivative exists and is zero on a set of measure 1. For each k, let $\mathcal{O}_k$ be the union of the $2^k - 1$ intervals which have been removed during the first k stages of the Cantor deletion process. Thus $C_k = [0, 1] \sim \mathcal{O}_k$. Define $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$. Then, by De Morgan's Identities, $\mathbf{C} = [0, 1] \sim \mathcal{O}$. We begin by defining φ on $\mathcal{O}$ and then we define it on $\mathbf{C}$.

[12] See page 19.

Fix a natural number k. Define φ on $\mathcal{O}_k$ to be the increasing function on $\mathcal{O}_k$ which is constant on each of its $2^k - 1$ open intervals and takes the $2^k - 1$ values

$$\{1/2^k, 2/2^k, 3/2^k, \ldots, [2^k - 1]/2^k\}.$$

Thus, on the single interval removed at the first stage of the deletion process, the prescription for φ is

$$\varphi(x) = 1/2 \text{ if } x \in (1/3, 2/3).$$

On the three intervals that are removed in the first two stages, the prescription for φ is

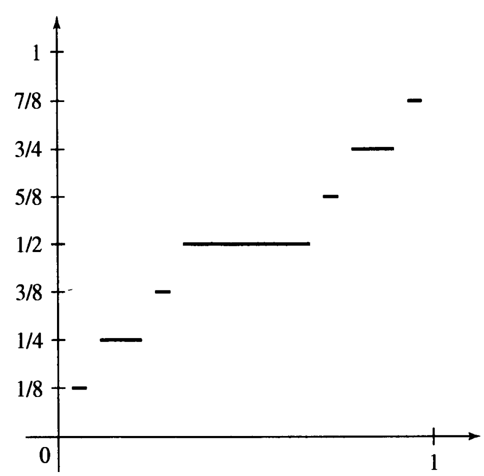
$$\varphi(x) = \begin{cases} 1/4 & \text{if } x \in (1/9,\ 2/9) \\ 2/4 & \text{if } x \in (3/9,\ 6/9) = (1/3, 2/3) \\ 3/4 & \text{if } x \in (7/9,\ 8/9) \end{cases}$$

We extend φ to all of [0, 1] by defining it on **C** as follows:

$$\varphi(0) = 0 \text{ and } \varphi(x) = \sup\left\{\varphi(t) \mid t \in \mathcal{O} \cap [0,\ x)\right\} \text{ if } x \in \mathbf{C} \sim \{0\}.$$

Proposition 20 *The Cantor-Lebesgue function φ is an increasing continuous function that maps* [0, 1] *onto* [0, 1]. *Its derivative exists on the open set $\mathcal{O}$, the complement in* [0, 1] *of the Cantor set,*

$$\varphi' = 0 \text{ on } \mathcal{O} \text{ while } m(\mathcal{O}) = 1.$$



The graph of the Cantor-Lebesgue function on $\mathcal{O}_3 = [0,\ 1] \sim C_3$

Proof Since φ is increasing on $\mathcal{O}$, its extension above to [0, 1] also is increasing. As for continuity, φ certainly is continuous at each point in $\mathcal{O}$ since for each such point belongs to an open interval on which it is constant. Now consider a point $x_0 \in \mathbf{C}$ with $x_0 \neq 0, 1$. Since the point x_0 belongs to **C** it is not a member of the $2^k - 1$ intervals removed in the first k stages of the removal process, whose union we denote by $\mathcal{O}_k$. Therefore, if k is sufficiently large, x_0 lies between two consecutive intervals in $\mathcal{O}_k$: choose a_k in the lower of these and b_k in the upper one. The function φ was defined to increase by $1/2^k$ across two consecutive intervals in $\mathcal{O}_k$. Therefore

$$a_k < x_0 < b_k \text{ and } \varphi(b_k) - \varphi(a_k) = 1/2^k.$$

Since k may be arbitrarily large, the function φ fails to have a jump discontinuity at x_0. For an increasing function, a jump discontinuity is the only possible type of discontinuity. Therefore φ is continuous at x_0. If x_0 is an endpoint of $[0, 1]$, a similar argument establishes continuity at x_0.

Since φ is constant on each of the intervals removed at any stage of the removal process, its derivative exists and equals 0 at each point in $\mathcal{O}$. Since **C** has measure zero, its complement in $[0, 1]$, $\mathcal{O}$, has measure 1. Finally, since $\varphi(0) = 0$, $\varphi(1) = 1$ and φ is increasing and continuous, we infer from the Intermediate Value Theorem that φ maps $[0, 1]$ onto $[0, 1]$. □

Proposition 21 *Let φ be the Cantor-Lebesgue function and define the function ψ on $[0, 1]$ by*

$$\psi(x) = \varphi(x) + x \textit{ for all } x \in [0, 1].$$

Then ψ is a strictly increasing continuous function that maps $[0, 1]$ onto $[0, 2]$,

(i) maps the Cantor set C onto a measurable set of positive measure and

(ii) maps a measurable set, a subset of the Cantor set, onto a nonmeasurable set.

Proof The function ψ is continuous since it is the sum of two continuous functions and is strictly increasing since it is the sum of an increasing and a strictly increasing function. Moreover, since $\psi(0) = 0$ and $\psi(1) = 2$, $\psi([0, 1]) = [0, 2]$. For $\mathcal{O} = [0, 1] \sim C$, we have the disjoint decomposition

$$[0, 1] = \mathbf{C} \cup \mathcal{O}$$

which ψ lifts to the disjoint decomposition

$$[0, 2] = \psi(\mathcal{O}) \cup \psi(\mathbf{C}). \tag{18}$$

A strictly increasing continuous function defined on an interval has a continuous inverse. Therefore $\psi(C)$ is closed and $\psi(\mathcal{O})$ is open, so both are measurable. We will show that $m(\psi(\mathcal{O})) = 1$ and therefore infer from (18) that $m(\psi(C)) = 1$ and thereby prove (i).

Let $\{I_k\}_{k=1}^{\infty}$ be an enumeration (in any manner) of the collection of intervals that are removed in the Cantor removal process. Thus $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Since φ is constant on each I_k, ψ maps I_k onto a translated copy of itself of the same length. Since ψ is one-to-one, the collection $\{\psi(I_k)\}_{k=1}^{\infty}$ is disjoint. By the countable additivity of measure,

$$m(\psi(\mathcal{O})) = \sum_{k=1}^{\infty} \ell(\psi(I_k)) = \sum_{k=1}^{\infty} \ell(I_k) = m(\mathcal{O}).$$

But $m(\mathbf{C}) = 0$ so that $m(\mathcal{O}) = 1$. Therefore $m(\psi(\mathcal{O})) = 1$ and hence, by (18), $m(\psi(\mathbf{C})) = 1$. We have established (i).

To verify (ii) we note that Vitali's Theorem tells us that $\psi(\mathbf{C})$ contains a set W, which is nonmeasurable. The set $\psi^{-1}(W)$ is measurable and has measure zero since it is a subset of the Cantor set. The set $\psi^{-1}(W)$ is a measurable subset of the Cantor set, which is mapped by ψ onto a nonmeasurable set. □

Proposition 22 *There is a measurable set, a subset of the Cantor set, that is not a Borel set.*

Proof The strictly increasing continuous function ψ defined on $[0, 1]$ that is described in the preceding proposition maps a measurable set A onto a nonmeasurable set. A strictly increasing continuous function defined on an interval maps Borel sets onto Borel sets (see Problem 47). Therefore the set A is not Borel since otherwise its image under ψ would be Borel and therefore would be measurable. □

PROBLEMS

34. Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.
35. Let f be an increasing function on the open interval I. For $x_0 \in I$ show that f is continuous at x_0 if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$, and $\lim_{n \to \infty}[f(b_n) - f(a_n)] = 0$.
36. Show that if f is any increasing function on $[0, 1]$ that agrees with the Cantor-Lebesgue function φ on the complement of the Cantor set, then $f = \varphi$ on all of $[0, 1]$.
37. Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?
38. Let the function $f\colon [a, b] \to \mathbf{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f maps a set of measure zero onto a set of measure *zero*. Show that f maps an F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.
39. Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0, 1] \sim F$ dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.
40. Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)
41. A nonempty subset X of $\mathbf{R}$ is called perfect provided it is closed and each neighborhood of any point in X contains infinitely many points of X. Show that the Cantor set is perfect. (Hint: The endpoints of all of the subintervals occurring in the Cantor construction belong to $\mathbf{C}$.)
42. Prove that every perfect subset X of $\mathbf{R}$ is uncountable. (Hint: If X is countable, construct a descending sequence of bounded, closed subsets of X whose intersection is empty.)
43. Use the preceding two problems to provide another proof of the uncountability of the Cantor set.
44. A subset A of $\mathbf{R}$ is said to be **nowhere dense** in $\mathbf{R}$ provided that for every open set $\mathcal{O}$ has an open subset that is disjoint from A. Show that the Cantor set is nowhere dense in $\mathbf{R}$.
45. Show that a strictly increasing function that is defined on an interval has a continuous inverse.
46. Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.)
47. Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.