Centre of mass :-

The centre of mass of a set of partic
is the point with respect to which the linear mo
of the set is zero.

where as linear moment of the set of partic
is given by

linear moment $= \sum m_i \vec{r}_i$, where $\vec{r}_i$
are position vectors of the masses m_i w.r.t. orig

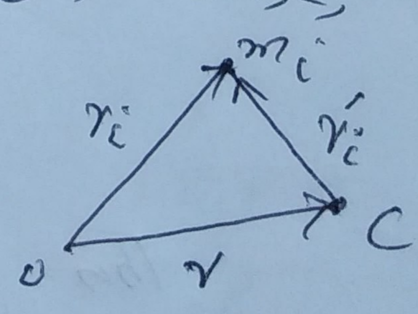
Let C be the centre of mass of a set of partic
$m_1, m_2, m_3, \ldots, m_n$ and let $\underline{r}$ be its position vector

Let the ~~points~~ masses $m_1, m_2, \ldots, m_n$ are
located at $P_1(r_1), P_2(r_2), \ldots, P_n(r_n)$.

If position vectors of the masses m_i ~~ar~~
with respect to C are r_i', then, as c.m
is the point with respect to which linear
moment is zero

So $\sum m_i \vec{r}_i' = 0$ (1) ; $i = 1, 2, \ldots, n$

From figure



$$\vec{r} + \vec{r}_i' = \vec{r}_i$$

$$\vec{r}_i' = \vec{r}_i - \vec{r}$$

using into (1)

$$\Sigma m_i(\vec{r}_i - \vec{r}) = 0$$

$$\Sigma m_i \vec{r}_i - \Sigma m_i \vec{r} = 0$$

$$\Sigma m_i \vec{r} = \Sigma m_i \vec{r}_i$$

$$(\Sigma m_i)\vec{r} = \Sigma m_i \vec{r}_i$$

$$\vec{r} = \frac{\Sigma m_i \vec{r}_i}{\Sigma m_i} \qquad (2)$$

Relation (2) gives the position vector of centre of mass of the set of masses m_i.

x———x

If we use $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ into eq (2)

we get

$$x = \frac{\Sigma m_i x_i}{\Sigma m_i}, \quad y = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

and $$z = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

Further if $m_1 = m_2 = \cdots = m_n$

then from (2)

$$\vec{r} = \frac{\Sigma \vec{r}_i}{n} \qquad (3)$$

or $$x = \frac{\Sigma x_i}{n}, \quad y = \frac{\Sigma y_i}{n}, \quad z = \frac{\Sigma z_i}{n}$$

This is called centroid of the points $P_1, P_2, \cdots P_n$. (As we study in geometry)

In the case of a set of two particles, we have from (2)

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$ and from (3)

$$\vec{r} = \frac{\vec{r}_1+\vec{r}_2}{2}$$ (a formula for mid point)

i, Centre of mass of a thin rod of unifo[rm] thickn[ess]

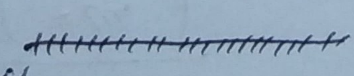
Let m be the mass of a uniform th[in] rod of length l, let one end of th[e] rod be taken as origin.

Let the rod be divided into n parts such that Δm_i is mass of it, ith part,

Then by def. of the c.o.m.

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$ (one dimension)

$$\Rightarrow \bar{x} = \lim_{\Delta m \to 0} \frac{\Sigma x_i \Delta m_i}{\Sigma m_i}$$

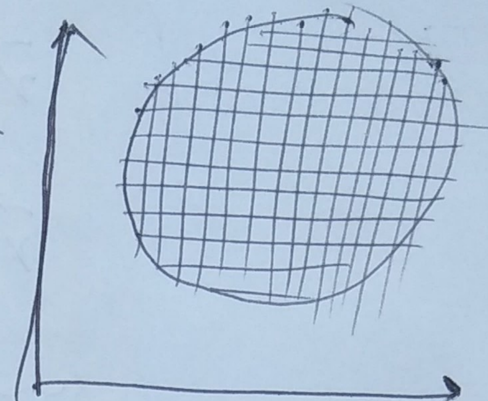


$$= \frac{\int_0^l x\,dm}{\int_0^l dm} = \frac{\int_0^l x \rho\,dx}{\int_0^l \rho\,dx}$$

$$= \frac{\frac{x^2}{2}\Big|_0^l}{\ldots} = \frac{l^2/2}{\ldots}$$

(ii) Centre of mass of a plane lamina.

Let m be the mass of the lamina and A its area. We divide the lamina into n parts (rectangles), so that Δm_i ~~and ΔA~~ is the mass of its ith part and $\vec{r}_i$ its position vector.



Then, by def of c.m.

$$\vec{r} = \frac{\sum m_i r_i}{\sum m_i}$$

$$\Rightarrow \quad \vec{r} = \frac{\sum r_i \Delta m_i}{\sum \Delta m_i}$$

As lamina can be considered as a collection of infinite no. of masses Δm_i, so

$$\vec{r} = \lim_{\Delta m_i \to 0} \frac{\sum \vec{r}_i \cdot \Delta m_i}{\sum \Delta m_i}$$

$$\Rightarrow \quad \bar{r} = \frac{\int_R \vec{r}\, dm}{\int_R dm}$$

$$\boxed{\begin{aligned} \rho &= \frac{m}{V} \\ m &= \rho V \\ m &= \rho A \end{aligned}}$$

$$= \frac{\int_R \vec{r} \rho\, dA}{\int_R \rho\, dA}$$

$$\bar{r} = \frac{\int_R \vec{r}\, dA}{\int_R dA} \quad \text{if density is constant.}$$

(iii) Centre of mass of a solid

To find the c.m. of a solid, let us divide the solid into n parts in such a way that mass of its ith part is Δm_i and each part is assu to be a rectangular ||piped shape

Then by def of c.m.

$$\bar{r} = \frac{\Sigma m_i r_i}{\Sigma m_i}$$

$$\Rightarrow \bar{r} = \frac{\Sigma r_i \Delta m_i}{\Sigma \Delta m_i}$$

Since solid can be considered as a collection of infinite small masses Δm_i.

So

$$\bar{r} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} r_i \Delta m_i}{\sum_{i=1}^{n} \Delta m_i}$$

$$= \lim_{n\to\infty} \frac{\Sigma r_i \rho \Delta V_i}{\Sigma \rho \Delta V_i}$$

$$= \frac{\int \rho r \, dV}{\int \rho \, dV}$$

$$= \int r \, dV / \int dV \quad \text{if } \rho = \text{Con}$$

In case of cartesian components,

from $\bar{r} = \dfrac{\int \rho r \, dV}{\int \rho \, dV} = \dfrac{\int \rho r \, dV}{m}$

$\Rightarrow \bar{x} = \dfrac{\int \rho x \, dx\,dy\,dz}{m}$, $\bar{y} = \dfrac{\int \rho y \, dx\,dy\,dz}{m}$

and $\bar{z} = \dfrac{\int \rho z \, dx\,dy\,dz}{m}$

or $m\bar{x} = \int \rho x \, dx\,dy\,dz$

$m\bar{y} = \int \rho y \, dx\,dy\,dz$

and $m\bar{z} = \int \rho z \, dx\,dy\,dz$
