PROBLEMS

24. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

25. Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

26. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

27. Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbf{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.
    (i) Show that m' is finitely additive, monotone, countably monotone, and possesses the excision property.
    (ii) Show that m' possesses the same continuity properties as Lebesgue measure.

28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

2.6 NONMEASURABLE SETS

We have defined what it means for a set to be measurable and studied properties of the collection of measurable sets. It is only natural to ask if, in fact, there are any sets that fail to be measurable. The answer is not at all obvious.

We know that if a set E has outer measure zero, then it is measurable, and since any subset of E also has outer measure zero, every subset of E is measurable. This is the best that can be said regarding the inheritance of measurability through the relation of set inclusion: we now show that if E is any set of real numbers with positive outer measure, then there are subsets of E that fail to be measurable.

Lemma 16 *Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$, is disjoint. Then $m(E) = 0$.*

Proof The translate of a measurable set is measurable. Thus, by the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda \in \Lambda} (\lambda + E)\right] = \sum_{\lambda \in \Lambda} m(\lambda + E). \tag{15}$$

Since both E and Λ are bounded sets, the set $\bigcup_{\lambda \in \Lambda}(\lambda + E)$ also is bounded and therefore has finite measure. Thus the left-hand side of (15) is finite. However, since measure is translation invariant, $m(\lambda + E) = m(E) > 0$ for each $\lambda \in \Lambda$. Thus, since the set Λ is countably infinite and the right-hand sum in (15) is finite, we must have $m(E) = 0$. □

For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to $\mathbf{Q}$, the set of rational numbers. It is easy to see that this is an equivalence relation, that is, it is reflexive, symmetric, and transitive. We call it the rational equivalence relation on E. For this relation, there is the disjoint decomposition of E into the collection of equivalence classes. By a **choice set** for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. We infer from the Axiom of Choice[10] that there are such choice sets. A choice set $\mathcal{C}_E$ is characterized by the following two properties:

(i) the difference of two points in $\mathcal{C}_E$ is not rational;

(ii) for each point x in E, there is a point c in $\mathcal{C}_E$ for which $x = c + q$, with q rational.

This first characteristic property of $\mathcal{C}_E$ may be conveniently reformulated as follows:

$$\text{For any set } \Lambda \subseteq \mathbf{Q},\ \{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda} \text{ is disjoint.} \tag{16}$$

Theorem 17 (Vitali) *Any set E of real numbers with positive outer measure contains a subset that fails to be measurable.*

Proof By the countable subadditivity of outer measure, we may suppose E is bounded. Let $\mathcal{C}_E$ be any choice set for the rational equivalence relation on E. We claim that $\mathcal{C}_E$ is not measurable. To verify this claim, we assume it is measurable and derive a contradiction.

Let Λ_0 be any bounded, countably infinite set of rational numbers. Since $\mathcal{C}_E$ is measurable, and, by (16), the collection of translates of $\mathcal{C}_E$ by members of Λ_0 is disjoint, it follows from Lemma 16 that $m(\mathcal{C}_E) = 0$. Hence, again using the translation invariance and the countable additivity of measure over countable disjoint unions of measurable sets,

$$m\left[\bigcup_{\lambda \in \Lambda_0} (\lambda + \mathcal{C}_E)\right] = \sum_{\lambda \in \Lambda_0} m(\lambda + \mathcal{C}_E) = 0.$$

To obtain a contradiction we make a special choice of Λ_0. Because E is bounded it is contained in some interval $[-b, b]$. We choose

$$\Lambda_0 = [-2b, 2b] \cap \mathbf{Q}.$$

Then Λ_0 is bounded, and is countably infinite since the rationals are countable and dense.[11] We claim that

$$E \subseteq \bigcup_{\lambda \in [-2b, 2b] \cap \mathbf{Q}} (\lambda + \mathcal{C}_E). \tag{17}$$

Indeed, by the second characteristic property of $\mathcal{C}_E$, if x belongs to E, there is a number c in the choice set $\mathcal{C}_E$ for which $x = c + q$ with q rational. But x and c belong to $[-b, b]$, so that q belongs to $[-2b, 2b]$. Thus the inclusion (17) holds. This is a contradiction because E, a set of positive outer measure, is not a subset of a set of measure zero. The assumption that $\mathcal{C}_E$ is measurable has led to a contradiction and thus it must fail to be measurable. □

[10] See page 5.

[11] See pages 12 and 14.

Theorem 18 *There are disjoint sets of real numbers A and B for which*

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

Proof We prove this by contradiction. Assume $m^*(A \cup B) = m^*(A) + m^*(B)$ for every disjoint pair of sets A and B. Then, by the very definition of measurable set, every set must be measurable. This contradicts the preceding theorem. □

PROBLEMS

29. (i) Show that rational equivalence defines an equivalence relation on any set.
    (ii) Explicitly find a choice set for the rational equivalence relation on **Q**.
    (iii) Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on **R**? Is this an equivalence relation on **Q**?
30. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.
31. Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.
32. Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?
33. Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which
$$m^*(E) = m^*(G), \text{ while } m^*(G \sim E) > 0.$$

2.7 THE CANTOR SET AND THE CANTOR-LEBESGUE FUNCTION

We have shown that a countable set has measure zero and a Borel set is Lebesgue measurable. These two assertions prompt the following two questions.

Question 1 If a set has measure zero, is it also countable?

Question 2 If a set is measurable, is it also Borel?

The answer to each of these questions is negative. In this section we construct a set called the Cantor set and a function called the Cantor-Lebesgue function. By studying these we answer the above two questions and later provide answers to other questions regarding finer properties of functions.

Consider the closed, bounded interval $I = [0, 1]$. The first step in the construction of the Cantor set is to subdivide I into three intervals of equal length $1/3$ and remove the interior of the middle interval, that is, we remove the interval $(1/3, 2/3)$ from the interval $[0, 1]$ to obtain the closed set C_1, which is the union of two disjoint closed intervals, each of length $1/3$:

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

We now repeat this "open middle one-third removal" on each of the two intervals in C_1 to obtain a closed set C_2, which is the union of 2^2 closed intervals, each of length $1/3^2$:

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$