Centre of mass of a union of two (any) disjoint sets of particles. (1)

Let $S_1 = \{m_1, m_2, \ldots, m_{n_1}\}$ and $S_2 = \{m_{n_1+1}, \ldots, m_n\}$ be two disjoint sets with known centre of masses $\bar{r}_1$ and $\bar{r}_2$ respectively.

Let $S = S_1 \cup S_2 = \{m_1, m_2, \ldots, m_{n_1}, m_{n_1+1}, \ldots, m_n\}$

If $\bar{r}$ is c.m. of the set S, then by definition

$$\bar{r} = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

$$= \frac{\sum_{i=1}^{n_1} m_i r_i + \sum_{i=n_1+1}^{n} m_i r_i}{\sum_{i=1}^{n_1} m_i + \sum_{i=n_1+1}^{n} m_i} \quad (1)$$

Now, since $\bar{r}_1$ is c.m. of $S_1 = \{m_1, m_2, \ldots, m_{n_1}\}$

so $$\bar{r}_1 = \frac{\sum_{i=1}^{n_1} m_i r_i}{\sum_{i=1}^{n_1} m_i} = \frac{\sum_{i=1}^{n_1} m_i r_i}{M_1}$$

$$\Rightarrow M_1 \bar{r}_1 = \sum_{i=1}^{n_1} m_i r_i, \text{ where } \sum_{i=1}^{n_1} m_i = M_1$$

Similarly $$M_2 \bar{r}_2 = \sum_{i=n_1+1}^{n} m_i r_i, \quad \sum_{i=n_1+1}^{n} m_i = M_2$$

By using these values in (1), we get

$$\bar{r} = \frac{M_1\bar{r}_1 + M_2\bar{r}_2}{M_1+M_2} \quad (2)$$ is c.m. of the union of two sets.

The result (2) can be extended to any number of union of disjoint sets.

i.e. If $S_1, S_2, S_3, \ldots$ are any no. of disjoint set, with masses $M_1, M_2, M_3, \ldots$, and $\bar{r}_1, \bar{r}_2, \bar{r}_3, \ldots$ as c.m. resp.

Then, c.m. of their union S is given by

$$\bar{r} = \frac{M_1\bar{r}_1 + M_2\bar{r}_2 + M_3\bar{r}_3 + \cdots}{M_1+M_2+M_3+\cdots}$$

To find the centre of mass of the set $S-S_1$ where c.m. of S and S_1 are known.

Let $S-S_1 = S_2$

Then $S_1 \cup S_2 = S$

Now if $\bar{r}_1, \bar{r}_2$ and $\bar{r}$ are centre of masses of the sets of particles S_1, S_2 and S

Then using the above result, we can write

$$\bar{r} = \frac{M_1\bar{r}_1 + M_2\bar{r}_2}{M_1+M_2}$$

or $\bar{r} = \dfrac{M_1\bar{r}_1 + M_2\bar{r}_2}{M}$ where $M_1 + M_2 = M \downarrow$ masses in

or $M\bar{r} = M_1\bar{r}_1 + M_2\bar{r}_2$

or $\bar{r}_2 = \dfrac{M\bar{r} - M_1\bar{r}_1}{M_2}$

or $\bar{r}_2 = \dfrac{M\bar{r} - M_1\bar{r}_1}{M - M_1}$ As req.

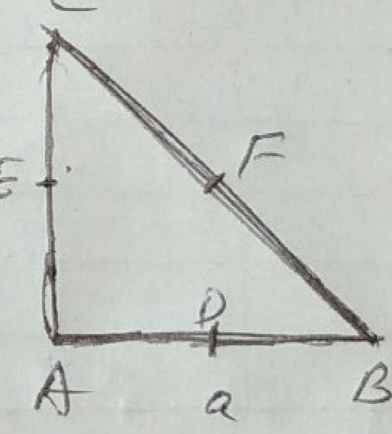
This result can also be extened for the difference $S - S_1 - S_2 - \cdots$, its c.m

is given by $\dfrac{M\bar{r} - M_1\bar{r}_1 - M_2\bar{r}_2 - \cdots}{M - M_1 - M_2 - \cdots}$

x———x

Q Find c.m. of an isosceles right triang formed with the help of uniform wire

Sol:- Let the Δ is given by the points A, B, C with A as origin and AB along x-axis, while AC along y-axis.



Let m be the mass of each side AB a

Since $|BC|^2 = |AB|^2 + |AC|^2 = a^2 + a^2 = 2a^2$

$|BC| = \sqrt{2}a$

So mass of the side BC is $\sqrt{2}m$

Further mid points of AB and AC are $D(\frac{a}{2}, 0)$, $E(0, \frac{a}{2})$, and mid of BC is $F(\frac{a}{2}, \frac{a}{2})$

Now we have to find c.m. of the ΔABC formed by three masses m, m and $\sqrt{2}m$ placed at the points D, E and F resp.

Therefore

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{m(\frac{a}{2}) + m(0) + \sqrt{2}m\frac{a}{2}}{m + m + \sqrt{2}m} = \frac{\frac{ma}{2}(1+\sqrt{2})}{m(1+1+\sqrt{2})}$$

$$\bar{x} = \frac{(1+\sqrt{2})a}{2(2+\sqrt{2})} = \frac{a}{2\sqrt{2}}$$

and

$$\bar{y} = \frac{m(0) + m(\frac{a}{2}) + \sqrt{2}m(\frac{a}{2})}{m + m + \sqrt{2}m} = \frac{\frac{ma}{2}(1+\sqrt{2})}{m(2+\sqrt{2})}$$

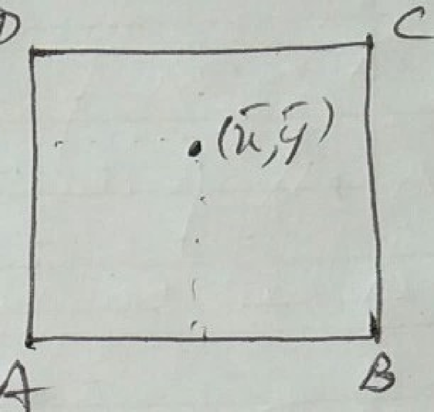
$$= \frac{a(1+\sqrt{2})}{2(2+\sqrt{2})} = \frac{a}{2\sqrt{2}}$$

Hence c.m. of $\Delta ABC = \left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$

Q.

Weights of 1, 2, 3, 4 lb, are placed at the corners A, B, C, D respectively of a square of side 8 inches. Find the distance of c.m. of the set of the weights from sides AB and AD.

Sol consider a square ABCD such that A is taken as origin, and AB, AD are along x-axis and y-axis respectively.



Then $A(0,0)$, $B(8,0)$, $C(8,8)$, $D(0,8)$.

Since masses 1, 2, 3 and 4 lbs are placed at the corners A, B, C and D respectively.

If $(\bar{x},\bar{y})$ is c.m. of the square,

Then
$$\bar{x} = \frac{m_1x_1+m_2x_2+m_3x_3+m_4x_4}{m_1+m_2+m_3+m_4}$$
$$= \frac{1(0)+2(8)+3(8)+4(0)}{1+2+3+4} = \frac{16+24}{10} = \frac{40}{10}$$
$$= 4 \text{ inches}$$

and
$$\bar{y} = \frac{m_1y_1+m_2y_2+m_3y_3+m_4y_4}{m_1+m_2+m_3+m_4}$$
$$= \frac{1(0)+2(0)+3(8)+4(8)}{1+2+3+4}$$
$$= \frac{24+32}{10} = \frac{56}{10} = 5.6 \text{ in.}$$

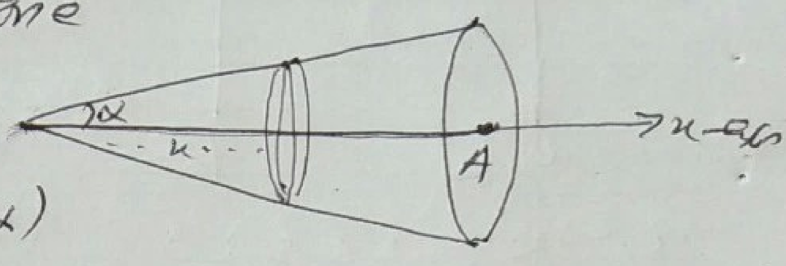
So distance from AD $= \bar{x} = 4$ in.

" " " AB $= \bar{y} = 5.6$ in

α

Q = Centre of mass of a right circular solid cone.

Let axis (OA) of the cone is along x-axis and O is origin (vertex)



A solid cone can be considered as a collection of slices of width δx, taken perp. to its axis.

If α is semivertical angle of the cone then radius of any one slice is given by $\dfrac{r}{x} = \tan\alpha$

$r = x\tan\alpha$, where x is distance of the slice from O.

So area of the slice $= \pi r^2$
$= \pi x^2 \tan^2\alpha$

And mass of the slice $= \rho \times$ volume
$= \rho \pi x^2 \tan^2\alpha \, \delta x$

Hence

$$\bar{x} = \frac{\int_0^h x\,dm}{\int_0^h dm}, \quad \bar{y} = 0$$

$$= \frac{\int_0^h x \pi \rho x^2 \tan^2\alpha \, dx}{\int_0^h \pi \rho x^2 \tan^2\alpha \, dx} = \frac{\int_0^h x^3 dx}{\int_0^h x^2 dx} = \frac{h^4/4}{h^3/3}$$

$= \frac{3h}{4}$