T, we obtain

$$\ddot{x} = \frac{\lambda}{ml} x$$

which is the equation of simple harmonic motion. If the initial conditions are assumed as $x(0) = x_0$, $\dot{x}(0) = 0$, then solution can be found as

$$x = x_0 \cos\left(\sqrt{\lambda/(ml)}\right) t$$

1.9.4 Potential energy of a linear spring

In addition to gravitational P.E., we come across another form of P.E. which is due to elastic deformation. As an example of such a P.E. we consider the P.E. of a spring. A linear spring consists of a mass m attached to an elastic bar. The elongation x of the spring is measured from its un-stretched position. The force in the spring when the particle has been stretched is proportional to the elongation and is given by $F = -kx$, where the constant $k (k > 0)$ is called *spring constant* (or *stiffness constant*). Writing $F = -(dv/dx)$, where $V(x)$ is the P.E. of the spring and solving the differential equation

$$\frac{dV}{dx} = kx$$

we obtain $V = (1/2)kx^2$. where we have chosen the zero-level of P.E. at $x = 0$

It is to be noted that F denotes the force exerted by the spring on the particle, and not the force of opposite sign which will be the force on the spring due to the particle.

1.10 Applications of the Principle of Conservation of Energy

In this section we will discuss some problems of mechanics, where principle of energy conservation is utilized to simplify calculational work or achieve better understanding of the problem. In order to solve a mechanical problem we usually begin with the equation(s) of motion which reduce to one or more second order differential equations. Then we solve this equation taking into account the initial or boundary conditions. This procedure may not always be simple or direct. When the system is known to be conservative, it is convenient start with the theorem (principle) of conservation of energy. We know that the K.E. is given by $mv^2/2$ with $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

The application ... equation of motion. For this ... Another integration will give the solution of the problem.

We will discuss the solution of some problems to illustrate the the application of the principle of energy conservation. Two of these problems are concerned with harmonic motion and the third with calculation of escape velocity. The harmonic oscillator is called simple when the displacements are infinitesimal, and it may be called plane or general when the same are finite.

1.10.1 The problem of simple harmonic oscillator

Usually the problem of simple harmonic motion is solved by solving the differential equation

$$\ddot{x}=-(k/m)x \quad \text{or} \quad \ddot{x}+\omega_0^2 x=0 \quad \text{where} \quad \omega_0^2=k/m.$$

We assume that simple or linear harmonic oscillator is a conservative system. There the total energy can be written as

$$E = T+V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

This relation gives

$$\begin{aligned}\dot{x} \equiv \frac{dx}{dt} &= \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2} \\ &= \pm\sqrt{\omega_0^2 A^2 - \omega_0^2 x^2} \\ &= \pm\omega_0\sqrt{A^2 - x^2} \end{aligned} \tag{1.10.1}$$

where we have put $2E/m = \omega_0^2 A^2$, $(A > 0)$ and used the relation $\omega_0^2 = (k/m)$.

From (1.10.1) on integration

$$\omega\int_{t_0}^{t} dt=\omega(t-t_0) = \int_{x_0}^{x}\frac{\pm dx}{\sqrt{A^2-x^2}} \tag{1.10.2}$$

where t_0 corresponds to x_0, the equilibrium position of the harmonic oscillator. To perform the integration in (1.10.2), we put $x = A\sin\theta$. Then

$$\omega_0(t-t_0) = \int_{\theta_0}^{\theta}\frac{A\cos\theta d\theta}{\cos\theta} = \int_{\theta_0}^{\theta} d\theta$$

$$\omega_0(t-t_0) = \theta-\theta_0 = \sin^{-1}\frac{x}{A} - \theta_0 \quad .$$

or

$$\sin^{-1}\frac{x}{A} = \omega_0(t-t_0) + \theta_0$$

$$x = A\sin(\omega_0(t-t_0)+\theta_0) \qquad (1.10.3)$$

where $\theta_0 = \sin^{-1}(x_0/A)$.

Equation (1.10.3) gives the solution of the problem of simple harmonic motion. It shows that the motion is periodic with period $\tau = 2\pi/\omega_0$.

1.10.2 Escape velocity

The force of gravitation due to the earth pulls the bodies towards the centre of the earth. That is the reason why bodies fall to the ground. If a body is projected away from the earth with sufficiently high speed, it may overcome the gravitational pull and escape into the outer space. The critical velocity at which this happens is called *escape velocity*. To calculate escape velocity we will make use of energy conservation. At the surface of the earth, the

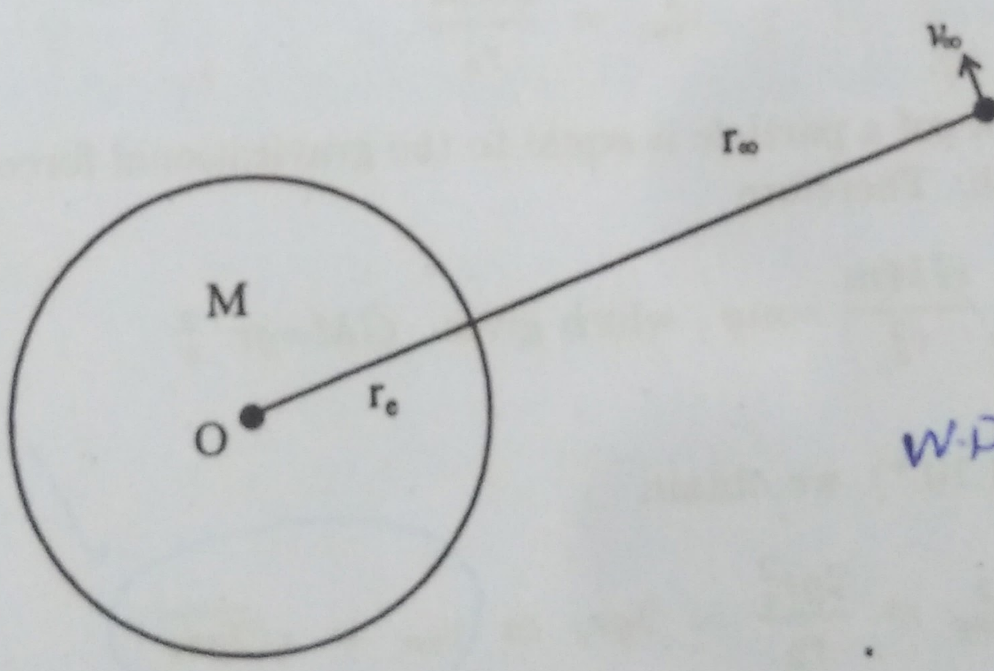

Figure 1.7:

potential energy is given by $V(r_e) = -(GMm/r_e)$, where r_e is the radius of the earth, M its mass and m the mass of the projectile. The potential energy will vanish at $r=\infty$. The projectile will escape from the gravitational pull of the earth when $V \to 0$. If must have some non-zero velocity and hence non-zero kinetic energy if it is to escape into the outer space. If $T(r=r_e)$ denotes the K.E. of the particle at the surface of the earth, then

$$T(r_e)+V(r_e) = \text{constant}\, C_0 \qquad (1.10.4)$$

If V_∞ denotes the non-zero velocity at in finite distance from the surface of the earth where gravitational P.E. has vanished, then $T(\infty) + V(\infty) = C_0$

or

$$\frac{1}{2}mv_\infty^2 + 0 = C_0 \quad (1.10.5)$$

From (1.10.4) and (1.10.5):

$$\frac{1}{2}mv_\infty^2 + 0 = \frac{1}{2}mv_0^2 + \left(-\frac{GMm}{r_e}\right)$$

or

$$v_\infty^2 = v_0^2 - \frac{2GM}{r_e}$$

Since $v_\infty \neq 0$, it follows that

$$v_0^2 \geq \frac{2GM}{r_e} \quad (1.10.6)$$

The critical condition is reached if we take the equality in (1.11.6). In that case $v_\infty = 0$ and the particle will reach infinity with zero velocity. The minimum escape velocity v_{esc} is then given by

$$v_{esc}^2 = \frac{2GM}{r_e} \quad (1.10.7)$$

Now the weight mg of a particle is equal to the gravitational force exerted on it by the earth. Therefore

$$\frac{GMm}{r_e^2} = mg \quad \text{which gives} \quad GM = gr_e^2$$

Substituting in (1.10.*), we obtain

$$v_{esc}^2 = \frac{2gr_e^2}{r_2} = 2gr_e \quad \text{or} \quad v_{esc} = \sqrt{2gr_e}$$

utting in the values $g \approx 9.81$ metres per sec ^{2}, $r_e \approx 6.4 \times 10^6$ metre, we btain $v_{esc} \approx 11.2$ kilometre per sec.

10.3 The harmonic motion of a plane pendulum

simple harmonic motion, the equation of motion viz. $\ddot{x} + kx = 0$ is inear differential equation and its solution(s) are periodic functions of e form $x = x_0 \sin(\omega t + \delta_1)$ or $x = x_0 \cos(\omega t + \delta_2)$ where $\omega = 2\pi/\nu$ is e angular frequency of the oscillator i...

The above equation of motion describes the motion of a pendulum when the swing of the pendulum is small. When there is no restriction on the amplitude of the pendulum, the corresponding equation of motion will be

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

or

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0 \tag{1.10.8}$$

where $\omega_0^2 = (g/\ell)$. To derive this equation we regard the oscillator as a particle with its weight $m\,g$ as the only force acting on it. Then the equation of motion will be

$$m\frac{d^2 s}{dt^2}\mathbf{t}_0 = -mg(\sin\theta)\mathbf{t}_0$$

where$s$is the arc element of the path of motion, $\mathbf{t}_0$ the unit tangent vector, and$\theta$the angle of inclination of the string with the vertical at time$t$. Then the equation of motion will be

$$m\frac{d^2 s}{dt^2}\mathbf{t}_0 = -mg\sin\theta\mathbf{t}_0$$

Using the result$s=l\theta$in the last equation we obtain (1.1.8). If we regard the oscillator as a sphere of finite radius, then we use rotational equation of motion $\mathbf{G} = \dot{\mathbf{L}}$, (see chapter 8) and obtain the same equation. Equation (1.10.9) is a non-linear ordinary differential equation of order 2.

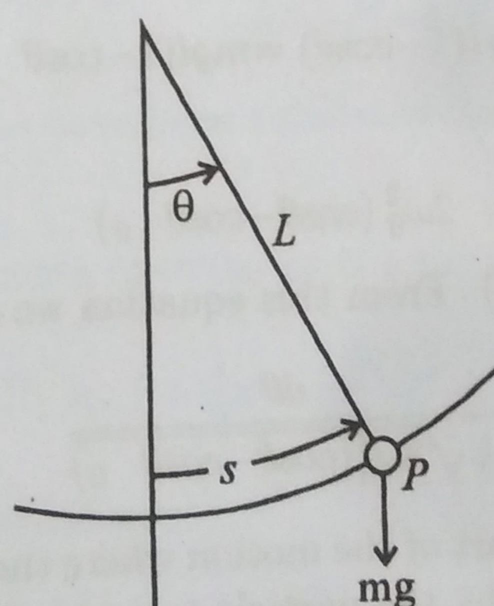

Figure 1.8:

One method to obtain its first integral is to multiply both sides by $2\dot{\theta}$and integrate w.r.t. t. This will give

$$\int \frac{d}{dt}(\dot{\theta}^2)dt + 2\omega_0^2 \int \sin\theta d\theta = c_1^2, \text{ constant}$$