PROBLEMS

16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).
17. Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$.
18. Let E have finite outer measure. Show that there is an F_σ set F and a G_δ set G such that
$$F \subseteq E \subseteq G \text{ and } m^*(F) = m^*(E) = m^*(G).$$
19. Let E have finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which
$$m^*(\mathcal{O} \sim E) > m^*(\mathcal{O}) - m^*(E).$$
20. (Lebesgue) Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a, b),
$$b - a = m^*((a, b) \cap E) + m^*((a, b) \sim E).$$
21. Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).
22. For any set A, define $m^{**}(A) \in [0, \infty]$ by
$$m^{**}(A) = \inf \left\{ m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open.} \right\}$$
How is this set function m^{**} related to outer measure m^*?
23. For any set A, define $m^{***}(A) \in [0, \infty]$ by
$$m^{***}(A) = \sup \left\{ m^*(F) \mid F \subseteq A, F \text{ closed.} \right\}$$
How is this set function m^{***} related to outer measure m^*?

2.5 COUNTABLE ADDITIVITY, CONTINUITY, AND THE BOREL-CANTELLI LEMMA

Definition *The restriction of the set function outer measure to the class of measurable sets is called* **Lebesgue measure**. *It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by*
$$m(E) = m^*(E).$$

The following proposition is of fundamental importance.

Proposition 13 *Lebesgue measure is countably additive, that is, if $\{E_k\}_{k=1}^\infty$ is a countable disjoint collection of measurable sets, then its union $\bigcup_{k=1}^\infty E_k$ also is measurable and*
$$m\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m(E_k).$$

Proof Proposition 7 tells us that $\bigcup_{k=1}^{\infty} E_k$ is measurable. According to Proposition 3, outer measure is countably subadditive. Thus

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m(E_k). \tag{9}$$

It remains to prove this inequality in the opposite directon. According to Proposition 6, for each natural number n,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k).$$

Since $\bigcup_{k=1}^{\infty} E_k$ contains $\bigcup_{k=1}^{n} E_k$, by the monotonicity of outer measure and the preceding equality,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{n} m(E_k) \text{ for each } n.$$

The left-hand side of this inequality is independent of n. Therefore

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{\infty} m(E_k). \tag{10}$$

From the inequalities (9) and (10) it follows that these are equalities. □

According to Proposition 1, the outer measure of an interval is its length while according to Proposition 2, outer measure is translation invariant. Therefore the preceding proposition completes the proof of the following theorem, which has been the principal goal of this chapter.

Theorem 14 *The set function Lebesgue measure, defined on the σ-algebra of Lebesgue measurable sets, assigns length to any interval, is translation invariant, and is countable additive.*

A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be **ascending** provided for each k, $E_k \subseteq E_{k+1}$, and said to be **descending** provided for each k, $E_{k+1} \subseteq E_k$.

Theorem 15 **(the Continuity of Measure)** *Lebesgue measure possesses the following continuity properties:*

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k). \tag{11}$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k). \tag{12}$$

Proof We first prove (i). If there is an index k_0 for which $m(A_{k_0}) = \infty$, then, by the monotonicity of measure, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$ and $m(A_k) = \infty$ for all $k \geq k_0$. Therefore (11) holds since each side equals ∞. It remains to consider the case that $m(A_k) < \infty$ for all k. Define $A_0 = \emptyset$ and then define $C_k = A_k \sim A_{k-1}$ for each $k \geq 1$. By construction, since the sequence $\{A_k\}_{k=1}^{\infty}$ is ascending,

$$\{C_k\}_{k=1}^{\infty} \text{ is disjoint and } \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k.$$

By the countable additivity of m,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} m(A_k \sim A_{k-1}). \tag{13}$$

Since $\{A_k\}_{k=1}^{\infty}$ is ascending, we infer from the excision property of measure that

$$\begin{aligned}\sum_{k=1}^{\infty} m(A_k \sim A_{k-1}) &= \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})] \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \\ &= \lim_{n \to \infty} [m(A_n) - m(A_0)].\end{aligned} \tag{14}$$

Since $m(A_0) = m(\emptyset) = 0$, (11) follows from (13) and (14).

To prove (ii) we define $D_k = B_1 \sim B_k$ for each k. Since the sequence $\{B_k\}_{k=1}^{\infty}$ is descending, the sequence $\{D_k\}_{k=1}^{\infty}$ is ascending. By part (i),

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k \to \infty} m(D_k).$$

According to De Morgan's Identities,

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \sim B_k] = B_1 \sim \bigcap_{k=1}^{\infty} B_k.$$

On the other hand, by the excision property of measure, for each k, since $m(B_k) < \infty$, $m(D_k) = m(B_1) - m(B_k)$. Therefore

$$m\left(B_1 \sim \bigcap_{k=1}^{\infty} B_k\right) = \lim_{n \to \infty} [m(B_1) - m(B_n)].$$

Once more using excision we obtain the equality (12). □

For a measurable set E, we say that a property holds **almost everywhere on** E, or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E \sim E_0$.

The Borel-Cantelli Lemma *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbf{R}$ belong to at most finitely many of the E_k's.*

Proof For each n, by the countable subadditivity of m,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Hence, by the continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore almost all $x \in \mathbf{R}$ fail to belong to $\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]$ and therefore belong to at most finitely many E_k's. □

The set function Lebesgue measure inherits the properties possessed by Lebesgue outer measure. For future reference we name some of these properties.

(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k).$$

(Monotonicity) If A and B are measurable sets and $A \subseteq B$, then

$$m(A) \leq m(B).$$

(Excision) If, moreover, $A \subseteq B$ and $m(A) < \infty$, then

$$m(B \sim A) = m(B) - m(A),$$

so that if $m(A) = 0$, then

$$m(B \sim A) = m(B).$$

(Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E,

$$m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$

Countable monotonicity is an amalgamation of the monotonicity and countable subadditivity properties of measure that is often invoked.

Remark *In our forthcoming study of Lebesgue integration it will be apparent that it is the countable additivity of Lebesgue measure that provides the Lebesgue integral with its decisive advantage over the Riemann integral.*