13. Show that (i) the translate of an F_σ set is also F_σ, (ii) the translate of a G_δ set is also G_δ, and (iii) the translate of a set of measure zero also has measure zero.
14. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.
15. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

2.4 OUTER AND INNER APPROXIMATION OF LEBESGUE MEASURABLE SETS

We now present two characterizations of measurability of a set, one based on inner approximation by closed sets and the other on outer approximation by open sets, which provide alternate angles of vision on measurability. These characterizations will be essential tools for our forthcoming study of approximation properties of measurable and integrable functions.

Measurable sets possess the following **excision property**: If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \sim A) = m^*(B) - m^*(A). \tag{7}$$

Indeed, by the measurability of A,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^C) = m^*(A) + m^*(B \sim A),$$

and hence, since $m^*(A) < \infty$, we have (7).

Theorem 11 *Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E.*
(Outer Approximation by Open Sets and G_δ Sets)

(i) *For each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \sim E) < \epsilon$.*
(ii) *There is a G_δ set G containing E for which $m^*(G \sim E) = 0$.*

(Inner Approximation by Closed Sets and F_σ Sets)

(iii) *For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \sim F) < \epsilon$.*
(iv) *There is an F_σ set F contained in E for which $m^*(E \sim F) = 0$.*

Proof We establish the equivalence of the measurability of E with each of the two outer approximation properties (i) and (ii). The remainder of the proof follows from De Morgan's Identities together with the observations that a set is measurable if and only if its complement is measurable, is open if and only if its complement is closed, and is F_σ if and only if its complement is G_δ.

Assume E is measurable. Let $\epsilon > 0$. First consider the case that $m^*(E) < \infty$. By the definition of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ which covers E and for which

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon.$$

Define $\mathcal{O} = \bigcup_{k=1}^\infty I_k$. Then $\mathcal{O}$ is an open set containing E. By the definition of the outer measure of $\mathcal{O}$,

$$m^*(\mathcal{O}) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon,$$

so that

$$m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

However, E is measurable and has finite outer measure. Therefore, by the excision property of measurable sets noted above,

$$m^*(\mathcal{O} \sim E) = m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

Now consider the case that $m^*(E) = \infty$. Then E may be expressed as the disjoint union of a countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer measure. By the finite measure case, for each index k, there is an open set $\mathcal{O}_k$ containing E_k for which $m^*(\mathcal{O}_k \sim E_k) < \epsilon/2^k$. The set $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$ is open, it contains E and

$$\mathcal{O} \sim E = \bigcup_{k=1}^{\infty} \mathcal{O}_k \sim E \subseteq \bigcup_{k=1}^{\infty} [\mathcal{O}_k \sim E_k].$$

Therefore

$$m^*(\mathcal{O} \sim E)) \leq \sum_{k=1}^{\infty} m^*(\mathcal{O}_k \sim E_k) < \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

Thus property (i) holds for E.

Now assume property (i) holds for E. For each natural number k, choose an open set $\mathcal{O}$ that contains E and for which $m^*(\mathcal{O}_k \sim E) < 1/k$. Define $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then G is a G_δ set that contains E. Moreover, since for each k, $G \sim E \subseteq \mathcal{O}_k \sim E$, by the monotonicity of outer measure,

$$m^*(G \sim E) \leq m^*(\mathcal{O}_k \sim E) < 1/k.$$

Therefore $m^*(G \sim E) = 0$ and so (ii) holds. Now assume property (ii) holds for E. Since a set of measure zero is measurable, as is a G_δ set, and the measurable sets are an algebra, the set

$$E = G \cap [G \sim E]^C$$

is measurable. □

The following property of measurable sets of finite outer measure asserts that such sets are "nearly" equal to the disjoint union of a finite number of open intervals.

Theorem 12 *Let E be a measurable set of finite outer measure. Then for each* $\epsilon > 0$, *there is a finite disjoint collection of open intervals* $\{I_k\}_{k=1}^{n}$ *for which if* $\mathcal{O} = \bigcup_{k=1}^{n} I_k$, *then*[8]

$$m^*(E \sim \mathcal{O}) + m^*(\mathcal{O} \sim E) < \epsilon.$$

Proof According to assertion (i) of Theorem 11, there is an open set $\mathcal{U}$ such that

$$E \subseteq \mathcal{U} \text{ and } m^*(\mathcal{U} \sim E) < \epsilon/2. \tag{8}$$

[8]For two sets A and B, the **symmetric difference** of A and B, which is denoted by $A \Delta B$, is defined to be the set $[A \sim B] \cup [B \sim A]$. With this notation the conclusion is that $m^*(E \Delta \mathcal{O}) < \epsilon$.

Since E is measurable and has finite outer measure, we infer from the excision property of outer measure that $\mathcal{U}$ also has finite outer measure. Every open set of real numbers is the disjoint union of a countable collection of open intervals.[9] Let $\mathcal{U}$ be the union of the countable disjoint collection of open intervals $\{I_k\}_{k=1}^{\infty}$. Each interval is measurable and its outer measure is its length. Therefore, by Proposition 6 and the monotonicity of outer measure, for each natural number n,

$$\sum_{k=1}^{n} \ell(I_k) = m^*\left(\bigcup_{k=1}^{n} I_k\right) \leq m^*(\mathcal{U}) < \infty.$$

The right-hand side of this inequality is independent of n. Therefore

$$\sum_{k=1}^{\infty} \ell(I_k) < \infty.$$

Choose a natural number n for which

$$\sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Define $\mathcal{O} = \bigcup_{k=1}^{n} I_k$. Since $\mathcal{O} \sim E \subseteq \mathcal{U} \sim E$, by the monotonicity of outer measure and (8),

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{U} \sim E) < \epsilon/2.$$

On the other hand, since $E \subseteq \mathcal{U}$,

$$E \sim \mathcal{O} \subseteq \mathcal{U} \sim \mathcal{O} = \bigcup_{k=n+1}^{\infty} I_k,$$

so that by the definition of outer measure,

$$m^*(E \sim \mathcal{O}) \leq \sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Thus

$$m^*(\mathcal{O} \sim E) + m^*(E \sim \mathcal{O}) < \epsilon.$$

□

Remark *A comment regarding assertion (i) in Theorem 11 is in order. By the definition of outer measure, for any bounded set E, regardless of whether or not it is measurable, and any* $\epsilon > 0$*, there is an open set* $\mathcal{O}$ *such that* $E \subseteq \mathcal{O}$ *and* $m^*(\mathcal{O}) < m^*(E) + \epsilon$ *and therefore* $m^*(\mathcal{O}) - m^*(E) < \epsilon$*. This does not imply that* $m^*(\mathcal{O} \sim E) < \epsilon$*, because the excision property*

$$m^*(\mathcal{O} \sim E) = m^*(\mathcal{O}) - m^*(E)$$

is false unless E is measurable (see Problem 19).

[9]See page 17.

PROBLEMS

16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).
17. Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$.
18. Let E have finite outer measure. Show that there is an F_σ set F and a G_δ set G such that

$$F \subseteq E \subseteq G \text{ and } m^*(F) = m^*(E) = m^*(G).$$

19. Let E have finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \sim E) > m^*(\mathcal{O}) - m^*(E).$$

20. (Lebesgue) Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \sim E).$$

21. Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).
22. For any set A, define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf \{m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open.}\}$$

How is this set function m^{**} related to outer measure m^*?

23. For any set A, define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup \{m^*(F) \mid F \subseteq A, F \text{ closed.}\}$$

How is this set function m^{***} related to outer measure m^*?

2.5 COUNTABLE ADDITIVITY, CONTINUITY, AND THE BOREL-CANTELLI LEMMA

Definition *The restriction of the set function outer measure to the class of measurable sets is called* **Lebesgue measure***. It is denoted by m, so that if E is a measurable set, its Lebesgue measure, $m(E)$, is defined by*

$$m(E) = m^*(E).$$

The following proposition is of fundamental importance.

Proposition 13 *Lebesgue measure is countably additive, that is, if $\{E_k\}_{k=1}^\infty$ is a countable disjoint collection of measurable sets, then its union $\bigcup_{k=1}^\infty E_k$ also is measurable and*

$$m\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m(E_k).$$