$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{1\leq k,i<\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \ell(I_{k,i})\right]$$

$$< \sum_{k=1}^{\infty}\left[m^*(E_k)+\epsilon/2^k\right]$$

$$= \left[\sum_{k=1}^{\infty} m^*(E_k)\right]+\epsilon.$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete. □

If $\{E_k\}_{k=1}^n$ is any finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k).$$

This **finite subadditivity** property follows from countable subadditivity by taking $E_k = \emptyset$ for $k > n$.

PROBLEMS

5. By using properties of outer measure, prove that the interval [0, 1] is not countable.
6. Let A be the set of irrational numbers in the interval [0, 1]. Prove that $m^*(A) = 1$.
7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

8. Let B be the set of rational numbers in the interval [0, 1], and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.
9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.
10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.3 THE σ-ALGEBRA OF LEBESGUE MEASURABLE SETS

Outer measure has four virtues: (i) it is defined for all sets of real numbers, (ii) the outer measure of an interval is its length, (iii) outer measure is countably subadditive, and (iv) outer measure is translation invariant. But outer measure fails to be countably additive. In fact, it is not even finitely additive (see Theorem 18): there are disjoint sets A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{3}$$

To ameliorate this fundamental defect we identify a σ-algebra of sets, called the Lebesgue measurable sets, which contains all intervals and all open sets and has the property that the restriction of the set function outer measure to the collection of Lebesgue measurable sets is countably additive. There are a number of ways to define what it means for a set to be measurable.[5] We follow an approach due to Constantine Carathéodory.

Definition *A set E is said to be* ***measurable*** *provided for any set A,*[6]

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C).$$

We immediately see one advantage possessed by measurable sets, namely, that the strict inequality (3) cannot occur if one of the sets is measurable. Indeed, if, say, A is measurable and B is any set disjoint from A, then

$$m^*(A \cup B) = m^*([A \cup B] \cap A)) + m^*([A \cup B)] \cap A^C) = m^*(A) + m^*(B).$$

Since, by Proposition 3, outer measure is finitely subadditive and $A = [A \cap E] \cup [A \cap E^C]$, we always have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C).$$

Therefore E is measurable if and only if for each set A we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^C). \tag{4}$$

This inequality trivially holds if $m^*(A) = \infty$. Thus it suffices to establish (4) for sets A that have finite outer measure.

Observe that the definition of measurability is symmetric in E and E^C, and therefore a set is measurable if and only if its complement is measurable. Clearly the empty-set $\emptyset$ and the set $\mathbf{R}$ of all real numbers are measurable.

Proposition 4 *Any set of outer measure zero is measurable. In particular, any countable set is measurable.*

Proof Let the set E have outer measure zero. Let A be any set. Since

$$A \cap E \subseteq E \text{ and } A \cap E^C \subseteq A,$$

by the monotonicity of outer measure,

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \cap E^C) \leq m^*(A).$$

Thus,

$$m^*(A) \geq m^*(A \cap E^C) = 0 + m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C),$$

and therefore E is measurable. □

[5] We should fully identify what we here call a measurable set as a Lebesgue measurable subset of the real line. A more general concept of measurable set will be studied in Part III. However, there will be no confusion in the first part of this book in simply using the adjective measurable.

[6] Recall that for a set E, by E^C we denote the set $\{x \in \mathbf{R} \mid x \notin E\}$, the **complement** of E in $\mathbf{R}$. We also denote E^C by $\mathbf{R} \sim E$. More generally, for two sets A and B, we let $A \sim B$ denote $\{a \in A \mid x \notin B\}$ and call it the **relative complement** of B in A.

Proposition 5 *The union of a finite collection of measurable sets is measurable.*

Proof As a first step in the proof, we show that the union of two measurable sets E_1 and E_2 is measurable. Let A be any set. First using the measurability of E_1, then the measurability of E_2, we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C)$$

$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C).$$

There are the following set identities:

$$[A \cap E_1^C] \cap E_2^C = A \cap [E_1 \cup E_2]^C$$

and

$$[A \cap E_1] \cup [A \cap E_1^C \cap E_2] = A \cap [E_1 \cup E_2].$$

We infer from these identities and the finite subadditivity of outer measure that

$$m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C)$$

$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^C)$$

$$\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C).$$

Thus $E_1 \cup E_2$ is measurable.

Now let $\{E_k\}_{k=1}^n$ be any finite collection of measurable sets. We prove the measurability of the union $\bigcup_{k=1}^n E_k$, for general n, by induction. This is trivial for $n = 1$. Suppose it is true for $n - 1$. Thus, since

$$\bigcup_{k=1}^{n} E_k = \left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n,$$

and we have established the measurability of the union of two measurable sets, the set $\bigcup_{k=1}^n E_k$ is measurable. □

Proposition 6 *Let A be any set and $\{E_k\}_{k=1}^n$ a finite disjoint collection of measurable sets. Then*

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

In particular,

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k).$$

Proof The proof proceeds by induction on n. It is clearly true for $n = 1$. Assume it is true for $n - 1$. Since the collection $\{E_k\}_{k=1}^n$ is disjoint,

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n = A \cap E_n$$

and

$$A \cap \left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^C = A \cap \left[\bigcup_{k=1}^{n-1} E_k\right].$$

Hence, by the measurability of E_n and the induction assumption,

$$m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{k=1}^{n-1} E_k\right]\right)$$

$$= m^*(A \cap E_n) + \sum_{k=1}^{n-1} m^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} m^*(A \cap E_k).$$

□

A collection of subsets of **R** is called an **algebra** provided it contains **R** and is closed with respect to the formation of complements and finite unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of finite intersections. We infer from Proposition 5, together with the measurability of the complement of a measurable set, that the collection of measurable sets is an algebra. It is useful to observe that the union of a countable collection of measurable sets is also the union of a countable disjoint collection of measurable sets. Indeed, let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets. Define $A'_1 = A_1$ and for each $k \geq 2$, define

$$A'_k = A_k \sim \bigcup_{i=1}^{k-1} A_i.$$

Since the collection of measurable sets is an algebra, $\{A'_k\}_{k=1}^{\infty}$ is a disjoint collection of measurable sets whose union is the same as that of $\{A_k\}_{k=1}^{\infty}$.

Proposition 7 *The union of a countable collection of measurable sets is measurable.*

Proof Let E be the union of a countable collection of measurable sets. As we observed above, there is a countable disjoint collection of measurable sets $\{E_k\}_{k=1}^{\infty}$ for which $E = \bigcup_{k=1}^{\infty} E_k$. Let A be any set. Let n be a natural number. Define $F_n = \bigcup_{k=1}^{n} E_k$. Since F_n is measurable and $F_n^C \supseteq E^C$,

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geq m^*(A \cap F_n) + m^*(A \cap E^C).$$

By Proposition 6,

$$m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Thus

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^C).$$

The left-hand side of this inequality is independent of n. Therefore

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*\left(A \cap E^C\right).$$

Hence, by the countable subadditivity of outer measure,

$$m^*(A) \geq m^*(A \cap E) + m^*\left(A \cap E^C\right).$$

Thus E is measurable. □

A collection of subsets of $\mathbf{R}$ is called an **σ-algebra** provided it contains $\mathbf{R}$ and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of countable intersections. The preceding proposition tells us that the collection of measurable sets is a σ-algebra.

Proposition 8 *Every interval is measurable.*

Proof As we observed above, the measurable sets are a σ-algebra. Therefore to show that every interval is measurable it suffices to show that every interval of the form (a, ∞) is measurable (see Problem 11). Consider such an interval. Let A be any set. We assume a does not belong to A. Otherwise, replace A by $A \sim \{a\}$, leaving the outer measure unchanged. We must show that

$$m^*(A_1) + m^*(A_2) \leq m^*(A), \tag{5}$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty).$$

By the definition of $m^*(A)$ as an infimum, to verify (5) it is necessary and sufficient to show that for any countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals that covers A,

$$m^*(A_1) + m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k). \tag{6}$$

Indeed, for such a covering, for each index k, define

$$I'_k = I_k \cap (-\infty, a) \text{ and } I''_k = I_k \cap (a, \infty)$$

Then I'_k and I''_k are intervals and

$$\ell(I_k) = \ell(I'_k) + \ell(I''_k).$$

Since $\{I'_k\}_{k=1}^{\infty}$ and $\{I''_k\}_{k=1}^{\infty}$ are countable collections of open, bounded intervals that cover A_1 and A_2, respectively, by the definition of outer measure,

$$m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I'_k) \text{ and } m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I''_k).$$

Therefore

$$m^*(A_1) + m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'').$$

$$= \sum_{k=1}^{\infty} [\ell(I_k') + \ell(I_k'')]$$

$$= \sum_{k=1}^{\infty} \ell(I_k).$$

Thus (6) holds and the proof is complete. □

Every open set is the disjoint union of a countable collection of open intervals.[7] We therefore infer from the two preceding propositions that every open set is measurable. Every closed set is the complement of an open set and therefore every closed set is measurable. Recall that a set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets and said to be an F_σ set provided it is the union of a countable collection of closed sets. We infer from Proposition 7 that every G_δ set and every F_σ set is measurable.

The intersection of all the σ-algebras of subsets of $\mathbf{R}$ that contain the open sets is a σ-algebra called the Borel σ-algebra; members of this collection are called **Borel sets**. The Borel σ-algebra is contained in every σ-algebra that contains all open sets. Therefore, since the measurable sets are a σ-algebra containing all open sets, every Borel set is measurable. We have established the following theorem.

Theorem 9 *The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains the σ-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.*

Proposition 10 *The translate of a measurable set is measurable.*

Proof Let E be a measurable set. Let A be any set and y be a real number. By the measurability of E and the translation invariance of outer measure,

$$m^*(A) = m^*(A - y) = m^*([A - y] \cap E) + m^*\left([A - y] \cap E^C\right)$$

$$= m^*(A \cap [E + y]) + m^*\left(A \cap [E + y]^C\right).$$

Therefore $E + y$ is measurable. □

PROBLEMS

11. Prove that if a σ-algebra of subsets of $\mathbf{R}$ contains intervals of the form (a, ∞), then it contains all intervals.
12. Show that every interval is a Borel set.

[7] See page 17.