PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.
2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.
3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.
4. A set function c, defined on all subsets of $\mathbf{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.

2.2 LEBESGUE OUTER MEASURE

Let I be a nonempty interval of real numbers. We define its length, $\ell(I)$, to be ∞ if I is unbounded and otherwise define its length to be the difference of its endpoints. For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure**[3] of A, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is **monotone** in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each natural number k, define $I_k = (c_k - \epsilon/2^{k+1},\ c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. Therefore

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$. Hence $m^*(E) = 0$.

Proposition 1 *The outer measure of an interval is its length.*

[3] There is a general concept of outer measure, which will be considered in Part III. The set function m^* is a particular example of this general concept, which is properly identified as Lebesgue outer measure on the real line. In Part I, we refer to m^* simply as outer measure.

Proof We begin with the case of a closed, bounded interval $[a, b]$. Let $\epsilon > 0$. Since the open interval $(a-\epsilon, b+\epsilon)$ contains $[a, b]$ we have $m^*([a, b]) \leq \ell((a-\epsilon, b+\epsilon)) = b-a+2\epsilon$. This holds for any $\epsilon > 0$. Therefore $m^*([a, b]) \leq b-a$. It remains to show that $m^*([a, b]) \geq b-a$. But this is equivalent to showing that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open, bounded intervals covering $[a, b]$, then

$$\sum_{k=1}^{\infty} \ell(I_k) \geq b-a. \tag{1}$$

By the Heine-Borel Theorem,[4] any collection of open intervals covering $[a, b]$ has a finite subcollection that also covers $[a, b]$. Choose a natural number n for which $\{I_k\}_{k=1}^{n}$ covers $[a, b]$. We will show that

$$\sum_{k=1}^{n} \ell(I_k) \geq b-a, \tag{2}$$

and therefore (1) holds. Since a belongs to $\bigcup_{k=1}^{n} I_k$, there must be one of the I_k's that contains a. Select such an interval and denote it by (a_1, b_1). We have $a_1 < a < b_1$. If $b_1 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq b_1 - a_1 > b-a.$$

Otherwise, $b_1 \in [a, b)$, and since $b_1 \notin (a_1, b_1)$, there is an interval in the collection $\{I_k\}_{k=1}^{n}$, which we label (a_2, b_2), distinct from (a_1, b_1), for which $b_1 \in (a_2, b_2)$; that is, $a_2 < b_1 < b_2$. If $b_2 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq (b_1 - a_1) + (b_2 - a_2) = b_2 - (a_2 - b_1) - a_1 > b_2 - a_1 > b - a.$$

We continue this selection process until it terminates, as it must since there are only n intervals in the collection $\{I_k\}_{k=1}^{n}$. Thus we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^{N}$ of $\{I_k\}_{k=1}^{n}$ for which

$$a_1 < a,$$

while

$$a_{k+1} < b_k \text{ for } 1 \leq k \leq N-1,$$

and, since the selection process terminated,

$$b_N > b.$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} \ell(I_k) &\geq \sum_{k=1}^{N} \ell((a_i, b_i)) \\ &= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \cdots + (b_1 - a_1) \\ &= b_N - (a_N - b_{N-1}) - \ldots - (a_2 - b_1) - a_1 \\ &> b_N - a_1 > b - a.\end{aligned}$$

[4]See page 18.

Thus the inequality (2) holds.

If I is any bounded interval, then given $\epsilon > 0$, there are two closed, bounded intervals J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2$$

while

$$\ell(I) - \epsilon < \ell(J_1) \text{ and } \ell(J_2) < \ell(I) + \epsilon.$$

By the equality of outer measure and length for closed, bounded intervals and the monotonicity of outer measure,

$$\ell(I) - \epsilon < \ell(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(J_2) < \ell(I) + \epsilon.$$

This holds for each $\epsilon > 0$. Therefore $\ell(I) = m^*(I)$.

If I is an unbounded interval, then for each natural number n, there is an interval $J \subseteq I$ with $\ell(J) = n$. Hence $m^*(I) \geq m^*(J) = \ell(J) = n$. This holds for each natural number n. Therefore $m^*(I) = \infty$. □

Proposition 2 *Outer measure is translation invariant, that is, for any set A and number y,*

$$m^*(A + y) = m^*(A).$$

Proof Observe that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of sets, then $\{I_k\}_{k=1}^{\infty}$ covers A if and only if $\{I_k + y\}_{k=1}^{\infty}$ covers $A + y$. Moreover, if each I_k is an open interval, then each $I_k + y$ is an open interval of the same length and so

$$\sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k + y).$$

The conclusion follows from these two observations. □

Proposition 3 *Outer measure is countably subadditive, that is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then*

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof If one of the E_k's has infinite outer measure, the inequality holds trivially. We therefore suppose each of the E_k's has finite outer measure. Let $\epsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open, bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} I_{k,i} \text{ and } \sum_{i=1}^{\infty} \ell(I_{k,i}) < m^*(E_k) + \epsilon/2^k.$$

Now $\{I_{k,i}\}_{1 \leq k, i \leq \infty}$ is a countable collection of open, bounded intervals that covers $\bigcup_{k=1}^{\infty} E_k$: the collection is countable since it is a countable collection of countable collections. Thus, by the definition of outer measure,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{1\leq k,i<\infty} \ell(I_{k,i}) = \sum_{k=1}^{\infty}\left[\sum_{i=1}^{\infty} \ell(I_{k,i})\right]$$
$$< \sum_{k=1}^{\infty}\left[m^*(E_k) + \epsilon/2^k\right]$$
$$= \left[\sum_{k=1}^{\infty} m^*(E_k)\right] + \epsilon.$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete. □

If $\{E_k\}_{k=1}^n$ is any finite collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} m^*(E_k).$$

This **finite subadditivity** property follows from countable subadditivity by taking $E_k = \emptyset$ for $k > n$.

PROBLEMS

5. By using properties of outer measure, prove that the interval [0, 1] is not countable.
6. Let A be the set of irrational numbers in the interval [0, 1]. Prove that $m^*(A) = 1$.
7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

8. Let B be the set of rational numbers in the interval [0, 1], and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.
9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.
10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.3 THE σ-ALGEBRA OF LEBESGUE MEASURABLE SETS

Outer measure has four virtues: (i) it is defined for all sets of real numbers, (ii) the outer measure of an interval is its length, (iii) outer measure is countably subadditive, and (iv) outer measure is translation invariant. But outer measure fails to be countably additive. In fact, it is not even finitely additive (see Theorem 18): there are disjoint sets A and B for which

$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{3}$$