CHAPTER 3

Lebesgue Measurable Functions

Contents

We devote this chapter to the study of measurable functions in order to lay the foundation for the study of the Lebesgue integral, which we begin in the next chapter. All continuous functions on a measurable domain are measurable, as are all monotone and step functions on a closed, bounded interval. Linear combinations of measurable functions are measurable. The pointwise limit of a sequence of measurable functions is measurable. We establish results regarding the approximation of measurable functions by simple functions and by continuous functions.

3.1 SUMS, PRODUCTS, AND COMPOSITIONS

All the functions considered in this chapter take values in the extended real numbers, that is, the set $\mathbf{R} \cup \{\pm\infty\}$. Recall that a property is said to hold **almost everywhere** (abbreviated a.e.) on a measurable set E provided it holds on $E \sim E_0$, where E_0 is a subset of E for which $m(E_0) = 0$.

Given two functions h and g defined on E, for notational brevity we often write "$h \leq g$ on E" to mean that $h(x) \leq g(x)$ for all $x \in E$. We say that a sequence of functions $\{f_n\}$ on E is increasing provided $f_n \leq f_{n+1}$ on E for each index n.

Proposition 1 *Let the function f have a measurable domain E. Then the following statements are equivalent:*

(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.
(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.
(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.
(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Each of these properties implies that for each extended real number c,

$$\text{the set } \{x \in E \mid f(x) = c\} \text{ is measurable.}$$

Proof Since the sets in (i) and (iv) are complementary in E, as are the sets in (ii) and (iii), and the complement in E of a measurable subset of E is measurable, (i) and (iv) are equivalent, as are (ii) and (iii).

Now (i) implies (ii), since

$$\{x \in E \mid f(x) \geq c\} = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > c - 1/k\},$$

and the intersection of a countable collection of measurable sets is measurable. Similarly, (ii) implies (i), since

$$\{x \in E \mid f(x) > c\} = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + 1/k\},$$

and the union of a countable collection of measurable sets is measurable.

Thus statements (i)–(iv) are equivalent. Now assume one, and hence all, of them hold. If c is a real number, $\{x \in E \mid f(x) = c\} = \{x \in E \mid f(x)) \geq c\} \cap \{x \in E \mid f(x) \leq c\}$, so $f^{-1}(c)$ is measurable since it is the intersection of two measurable sets. On the other hand, if c is infinite, say $c = \infty$,

$$\{x \in E \mid f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in E \mid f(x) > k\}$$

so $f^{-1}(\infty)$ is measurable since it is the intersection of a countable collection of measurable sets. □

Definition *An extended real-valued function f defined on E is said to be* **Lebesgue measurable**, *or simply* **measurable**, *provided its domain E is measurable and it satisfies one of the four statements of Proposition 1.*

Proposition 2 *Let the function f be defined on a measurable set E. Then f is measurable if and only if for each open set $\mathcal{O}$, the inverse image of $\mathcal{O}$ under f, $f^{-1}(\mathcal{O}) = \{x \in E \mid f(x) \in \mathcal{O}\}$, is measurable.*

Proof If the inverse image of each open set is measurable, then since each interval (c, ∞) is open, the function f is measurable. Conversely, suppose f is measurable. Let $\mathcal{O}$ be open. Then[1] we can express $\mathcal{O}$ as the union of a countable collection of open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ where each I_k may be expressed as $B_k \cap A_k$, where $B_k = (-\infty, b_k)$ and $A_k = (a_k, \infty)$. Since f is a measurable function, each $f^{-1}(B_k)$ and $f^{-1}(A_k)$ are measurable sets. On the other hand, the measurable sets are a σ-algebra and therefore $f^{-1}(\mathcal{O})$ is measurable since

$$f^{-1}(\mathcal{O}) = f^{-1}\left[\bigcup_{k=1}^{\infty} B_k \cap A_k\right] = \bigcup_{k=1}^{\infty} f^{-1}(B_k) \cap f^{-1}(A_k).$$ □

The following proposition tells us that the most familiar functions from elementary analysis, the continuous functions, are measurable.

Proposition 3 *A real-valued function that is continuous on its measurable domain is measurable.*

[1]See page 17.

Proof Let the function f be continuous on the measurable set E. Let $\mathcal{O}$ be open. Since f is continuous, $f^{-1}(\mathcal{O}) = E \cap \mathcal{U}$, where $\mathcal{U}$ is open.[2] Thus $f^{-1}(\mathcal{O})$, being the intersection of two measurable sets, is measurable. It follows from the preceding proposition that f is measurable. □

A real-valued function that is either increasing or decreasing is said to be monotone. We leave the proof of the next proposition as an exercise (see Problem 24).

Proposition 4 *A monotone function that is defined on an interval is measurable.*

Proposition 5 *Let f be an extended real-valued function on E.*

(i) If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.
(ii) For a measurable subset D of E, f is measurable on E if and only if the restrictions of f to D and $E \sim D$ are measurable.

Proof First assume f is measurable. Define $A = \{x \in E \mid f(x) \neq g(x)\}$. Observe that

$$\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\} \cup \Big[\{x \in E \mid f(x) > c\} \cap [E \sim A]\Big]$$

Since $f = g$ a.e. on E, $m(A) = 0$. Thus $\{x \in A \mid g(x) > c\}$ is measurable since it is a subset of a set of measure zero. The set $\{x \in E \mid f(x) > c\}$ is measurable since f is measurable on E. Since both E and A are measurable and the measurable sets are an algebra, the set $\{x \in E \mid g(x) > c\}$ is measurable. To verify (ii), just observe that for any c,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in E \sim D \mid f(x) > c\}$$

and once more use the fact that the measurable sets are an algebra. □

The sum $f + g$ of two measurable extended real-valued functions f and g is not properly defined at points at which f and g take infinite values of opposite sign. Assume f and g are finite a.e. on E. Define E_0 to be the set of points in E at which both f and g are finite. If the restriction of $f + g$ to E_0 is measurable, then, by the preceding proposition, any extension of $f + g$, as an extended real-valued function, to all of E also is measurable. This is the sense in which we consider it unambiguous to state that the sum of two measurable functions that are finite a.e. is measurable. Similar remarks apply to products. The following proposition tells us that standard algebraic operations performed on measurable functions that are finite a.e. again lead to measurable functions

Theorem 6 *Let f and g be measurable functions on E that are finite a.e. on E.*

(Linearity) For any α and β,

$$\alpha f + \beta g \text{ is measurable on } E.$$

(Products)

$$fg \text{ is measurable on } E.$$

[2]See page 25.

Proof By the above remarks, we may assume f and g are finite on all of E. If $\alpha = 0$, then the function αf also is measurable. If $\alpha \neq 0$, observe that for a number c,

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) > c/\alpha\} \text{ if } \alpha > 0$$

and

$$\{x \in E \mid \alpha f(x) > c\} = \{x \in E \mid f(x) < c/\alpha\} \text{ if } \alpha < 0.$$

Thus the measurability of f implies the measurability of αf. Therefore to establish linearity it suffices to consider the case that $\alpha = \beta = 1$.

For $x \in E$, if $f(x) + g(x) < c$, then $f(x) < c - g(x)$ and so, by the density of the set of rational numbers $\mathbf{Q}$ in $\mathbf{R}$, there is a rational number q for which

$$f(x) < q < c - g(x).$$

Hence

$$\{x \in E \mid f(x) + g(x) < c\} = \bigcup_{q \in \mathbf{Q}} \{x \in E \mid g(x) < c - q\} \cap \{x \in E \mid f(x) < q\}.$$

The rational numbers are countable. Thus $\{x \in E \mid f(x) + g(x) < c\}$ is measurable, since it is the union of a countable collection of measurable sets. Hence $f + g$ is measurable.

To prove that the product of measurable functions is measurable, first observe that

$$fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

Thus, since we have established linearity, to show that the product of two measurable functions is measurable it suffices to show that the square of a measurable function is measurable. For $c \geq 0$,

$$\{x \in E \mid f^2(x) > c\} = \{x \in E \mid f(x) > \sqrt{c}\} \cup \{x \in E \mid f(x) < -\sqrt{c}\}$$

while for $c < 0$,

$$\{x \in E \mid f^2(x) > c\} = E.$$

Thus f^2 is measurable. □

Many of the properties of functions considered in elementary analysis, including continuity and differentiability, are preserved under the operation of composition of functions. However, the composition of measurable functions may not be measurable.

Example There are two measurable real-valued functions, each defined on all of $\mathbf{R}$, whose composition fails to be measurable. By Lemma 21 of Chapter 2, there is a continuous, strictly increasing function ψ defined on $[0, 1]$ and a measurable subset A of $[0, 1]$ for which $\psi(A)$ is nonmeasurable. Extend ψ to a continuous, strictly increasing function that maps $\mathbf{R}$ onto $\mathbf{R}$. The function ψ^{-1} is continuous and therefore is measurable. On the other hand, A is a measurable set and so its characteristic function χ_A is a measurable function. We claim that the composition $f = \chi_A \circ \psi^{-1}$ is not measurable. Indeed, if I is any open interval containing 1 but not 0, then its inverse image under f is the nonmeasurable set $\psi(A)$.

Despite the setback imposed by this example, there is the following useful proposition regarding the preservation of measurability under composition (also see Problem 11).

Proposition 7 *Let g be a measurable real-valued function defined on E and f a continuous real-valued function defined on all of* **R**. *Then the composition $f \circ g$ is a measurable function on E.*

Proof According to Proposition 2, a function is measurable if and only if the inverse image of each open set is measurable. Let $\mathcal{O}$ be open. Then

$$(f \circ g)^{-1}(\mathcal{O}) = g^{-1}(f^{-1}(\mathcal{O})).$$

Since f is continuous and defined on an open set, the set $\mathcal{U} = f^{-1}(\mathcal{O})$ is open.[3] We infer from the measurability of the function g that $g^{-1}(\mathcal{U})$ is measurable. Thus the inverse image $(f \circ g)^{-1}(\mathcal{O})$ is measurable and so the composite function $f \circ g$ is measurable. □

An immediate important consequence of the above composition result is that if f is measurable with domain E, then $|f|$ is measurable, and indeed

$$|f|^p \text{ is measurable with the same domain } E \text{ for each } p > 0.$$

For a finite family $\{f_k\}_{k=1}^n$ of functions with common domain E, the function

$$\max\{f_1, \ldots, f_n\}$$

is defined on E by

$$\max\{f_1, \ldots, f_n\}(x) = \max\{f_1(x), \ldots, f_n(x)\} \text{ for } x \in E.$$

The function $\min\{f_1, \ldots, f_n\}$ is defined the same way.

Proposition 8 *For a finite family $\{f_k\}_{k=1}^n$ of measurable functions with common domain E, the functions* $\max\{f_1, \ldots, f_n\}$ *and* $\min\{f_1, \ldots, f_n\}$ *also are measurable.*

Proof For any c, we have

$$\{x \in E \mid \max\{f_1, \ldots, f_n\}(x) > c\} = \bigcup_{k=1}^{n} \{x \in E \mid f_k(x) > c\}$$

so this set is measurable since it is the finite union of measurable sets. Thus the function $\max\{f_1, \ldots, f_n\}$ is measurable. A similar argument shows that the function $\min\{f_1, \ldots, f_n\}$ also is measurable. □

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[3]See page 25.

For a function f defined on E, we have the associated functions $|f|$, f^+, and f^- defined on E by

$$|f|(x) = \max\{f(x), -f(x)\}, \; f^+(x) = \max\{f(x), 0\}, \; f^-(x) = \max\{-f(x), 0\}.$$

If f is measurable on E, then, by the preceding proposition, so are the functions $|f|$, f^+, and f^-. This will be important when we study integration since the expression of f as the difference of two nonnegative functions,

$$f = f^+ - f^- \text{ on } E,$$

plays an important part in defining the Lebesgue integral.

PROBLEMS

1. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?
2. Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ if and only if its restrictions to D and E are measurable. Is the same true if "measurable" is replaced by "continuous"?
3. Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?
4. Suppose f is a real-valued function on $\mathbf{R}$ such that $f^{-1}(c)$ is measurable for each number c. Is f necessarily measurable?
5. Suppose the function f is defined on a measurable set E and has the property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Is f necessarily measurable?
6. Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined on $\mathbf{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.
7. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable. (Hint: The collection of sets A that have the property that $f^{-1}(A)$ is measurable is a σ-algebra.)
8. (Borel measurability) A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each c, the set $\{x \in E \mid f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace "(Lebesgue) measurable set" by "'Borel set." Show that: (i) every Borel measurable function is Lebesgue measurable; (ii) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set; (iii) if f and g are Borel measurable, so is $f \circ g$; and (iv) if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.
9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?
10. Suppose f and g are real-valued functions defined on all of $\mathbf{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?
11. Let f be a measurable function and g be a one-to-one function from $\mathbf{R}$ onto $\mathbf{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable. (Hint: Examine Problem 38 in Chapter 2.)

3.2 SEQUENTIAL POINTWISE LIMITS AND SIMPLE APPROXIMATION

For a sequence $\{f_n\}$ of functions with common domain E and a function f on E, there are several distinct ways in which it is necessary to consider what it means to state that

"the sequence $\{f_n\}$ converges to f."

In this chapter we consider the concepts of pointwise convergence and uniform convergence, which are familiar from elementary analysis. In later chapters we consider many other modes of convergence for a sequence of functions.

Definition *For a sequence $\{f_n\}$ of functions with common domain E, a function f on E and a subset A of E, we say that*

(i) The sequence $\{f_n\}$ converges to f pointwise on A provided

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in A.$$

(ii) The sequence $\{f_n\}$ converges to f pointwise a.e. on A provided it converges to f pointwise on $A \sim B$, where $m(B) = 0$.

(iii) The sequence $\{f_n\}$ converges to f uniformly on A provided for each $\epsilon > 0$, there is an index N for which

$$|f - f_n| < \epsilon \text{ on } A \text{ for all } n \geq N.$$

When considering sequences of functions $\{f_n\}$ and their convergence to a function f, we often implicitly assume that all of the functions have a common domain. We write "$\{f_n\} \to f$ pointwise on A" to indicate the sequence $\{f_n\}$ converges to f pointwise on A and use similar notation for uniform convergence.

The pointwise limit of continuous functions may not be continuous. The pointwise limit of Riemann integrable functions may not be Riemann integrable. The following proposition is the first indication that the measureable functions have much better stability properties.

Proposition 9 *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f. Then f is measurable.*

Proof Let E_0 be a subset of E for which $m(E_0) = 0$ and $\{f_n\}$ converges to f pointwise on $E \sim E_0$. Since $m(E_0) = 0$, it follows from Proposition 5 that f is measurable if and only if its restriction to $E \sim E_0$ is measurable. Therefore, by possibly replacing E by $E \sim E_0$, we may assume the sequence converges pointwise on all of E.

Fix a number c. We must show that $\{x \in E \mid f(x) < c\}$ is measurable. Observe that for a point $x \in E$, since $\lim_{n\to\infty} f_n(x) = f(x)$,

$$f(x) < c$$

if and only if

there are natural numbers n and k for which $f_j(x) < c - 1/n$ for all $j \geq k$.

But for any natural numbers n and j, since the function f_j is measurable, the set $\{x \in E \mid f_j(x) < c - 1/n\}$ is measurable. Therefore, for any k, the intersection of the countably collection of measureable sets

$$\bigcap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\}$$

also is measurable. Consequently, since the union of a countable collection of measurable sets is measurable,

$$\{x \in E \mid f(x) < c\} = \bigcup_{1 \le k, n < \infty} \left[\bigcap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\} \right]$$

is measurable. □

If A is any set, the **characteristic function** of A, χ_A, is the function on **R** defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

It is clear that the function χ_A is measurable if and only if the set A is measurable. Thus the existence of a nonmeasurable set implies the existence of a nonmeasurable function. Linear combinations of characteristic functions of measurable sets play a role in Lebesgue integration similar to that played by step functions in Riemann integration, and so we name these functions.

Definition *A real-valued function φ defined on a measurable set E is called* ***simple*** *provided it is measurable and takes only a finite number of values.*

We emphasize that a simple function only takes real values. Linear combinations and products of simple functions are simple since each of them takes on only a finite number of values. If φ is simple, has domain E and takes the distinct values $c_1, \ldots, c_n$, then

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\}.$$

This particular expression of φ as a linear combination of characteristic functions is called the **canonical representation of the simple function** φ.

The Simple Approximation Lemma *Let f be a measurable real-valued function on E. Assume f is bounded on E, that is, there is an $M \ge 0$ for which $|f| \le M$* on E. *Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on E which have the following approximation properties:*

$$\varphi_\epsilon \le f \le \psi_\epsilon \text{ and } 0 \le \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } E.$$

Proof Let (c, d) be an open, bounded interval that contains the image of E, $f(E)$, and

$$c = y_0 < y_1 < \ldots < y_{n-1} < y_n = d$$

be a partition of the closed, bounded interval $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$. Defne

$$I_k = [y_{k-1}, y_k) \text{ and } E_k = f^{-1}(I_k) \text{ for } 1 \leq k \leq n.$$

Since each I_k is an interval and the function f is measurable, each set E_k is measurable. Define the simple functions φ_ϵ and ψ_ϵ on E by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}.$$

Let x belong to E. Since $f(E) \subseteq (c, d)$, there is a unique $k, 1 \leq k \leq n$, for which $y_{k-1} \leq f(x) < y_k$ and therefore

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x).$$

But $y_k - y_{k-1} < \epsilon$, and therefore φ_ϵ and ψ_ϵ have the required approximation properties. □

To the several characterizations of measurable functions that we already established, we add the following one.

The Simple Approximation Theorem *An extended real-valued function f on a measurable set E is measurable if and only if there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that*

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n.$$

If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof Since each simple function is measurable, Proposition 9 tells us that a function is measurable if it is the pointwise limit of a sequence of simple functions. It remains to prove the converse.

Assume f is measurable. We also assume $f \geq 0$ on E. The general case follows by expressing f as the difference of nonnegative measurable functions (see Problem 23). Let n be a natural number. Define $E_n = \{x \in E \mid f(x) \leq n.\}$ Then E_n is a measurable set and the restriction of f to E_n is a nonnegative bounded measurable function. By the Simple Approximation Lemma, applied to the restriction of f to E_n and with the choice of $\epsilon = 1/n$, we may select simple functions φ_n and ψ_n defined on E_n which have the following approximation properties:

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Observe that

$$0 \leq \varphi_n \leq f \text{ and } 0 \leq f - \varphi_n \leq \psi_n - \varphi_n < 1/n \text{ on } E_n. \tag{1}$$

Extend φ_n to all of E by setting $\varphi_n(x) = n$ if $f(x) > n$. The function φ_n is a simple function defined on E and $0 \le \varphi_n \le f$ on E. We claim that the sequence $\{\psi_n\}$ converges to f pointwise on E. Let x belong to E.

Case 1: Assume $f(x)$ is finite. Choose a natural number N for which $f(x) < N$. Then

$$0 \le f(x) - \varphi_n(x) < 1/n \text{ for } n \ge N,$$

and therefore $\lim_{n\to\infty} \psi_n(x) = f(x)$.

Case 2: Assume $f(x) = \infty$. Then $\varphi_n(x) = n$ for all n, so that $\lim_{n\to\infty} \varphi_n(x) = f(x)$.

By replacing each φ_n with $\max\{\varphi_1, \ldots, \varphi_n\}$ we have $\{\varphi_n\}$ increasing. □

PROBLEMS

12. Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing and each of these sequences converges to f uniformly on E.
13. A real-valued measurable function is said to be *semisimple* provided it takes only a countable number of values. Let f be any measurable function on E. Show that there is a sequence of semisimple functions $\{f_n\}$ on E that converges to f uniformly on E.
14. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \sim F) < \epsilon$.
15. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set F contained in E and a sequence $\{\varphi_n\}$ of simple functions on E such that $\{\varphi_n\} \to f$ uniformly on F and $m(E \sim F) < \epsilon$. (Hint: See the preceding problem.)
16. Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

(Hint: Use Theorem 12 of Chapter 2.)

17. Let I be a closed, bounded interval and ψ a simple function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

(Hint: Use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

18. Let I be a closed, bounded interval and f a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \sim F) < \epsilon.$$

19. Show that the sum and product of two simple functions are simple as are the max and the min.