CHAPTER 4

Lebesgue Integration

Contents

We now turn to our main object of interest in Part I, the Lebesgue integral. We define this integral in four stages. We first define the integral for simple functions over a set of finite measure. Then for bounded measurable functions f over a set of finite measure, in terms of integrals of upper and lower approximations of f by simple functions. We define the integral of a general nonnegative measurable function f over E to be the supremum of the integrals of lower approximations of f by bounded measurable functions that vanish outside a set of finite measure; the integral of such a function is nonnegative, but may be infinite. Finally, a general measurable function is said to be integrable over E provided $\int_E |f| < \infty$. We prove that linear combinations of integrable functions are integrable and that, on the class of integrable functions, the Lebesgue integral is a monotone, linear functional. A principal virtue of the Lebesgue integral, beyond the extent of the class of integrable functions, is the availability of quite general criteria which guarantee that if a sequence of integrable functions $\{f_n\}$ converge pointwise almost everywhere on E to f, then

$$\lim_{n\to\infty} \int_E f_n = \int_E \left[\lim_{n\to\infty} f_n\right] \equiv \int_E f.$$

We refer to that as passage of the limit under the integral sign. Based on Egoroff's Theorem, a consequence of the countable additivity of Lebesgue measure, we prove four theorems that provide criteria for justification of this passage: the Bounded Convergence Theorem, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem.

4.1 THE RIEMANN INTEGRAL

We recall a few definitions pertaining to the Riemann integral. Let f be a bounded real-valued function defined on the closed, bounded interval $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Define the **lower and upper Darboux sums** for f with respect to P, respectively, by

$$L(f, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}),$$

where,[1] for $1 \leq i \leq n$,

$$m_i = \inf \left\{ f(x) \mid x_{i-1} < x < x_i \right\} \text{ and } M_i = \sup \left\{ f(x) \mid x_{i-1} < x < x_i \right\}.$$

We then define the **lower and upper Riemann integrals** of f over $[a, b]$, respectively, by

$$(R)\underline{\int_a^b} f = \sup \left\{ L(f, P) \mid P \text{ a partition of } [a, b] \right\}$$

and

$$(R)\overline{\int_a^b} f = \inf \left\{ U(f, P) \mid P \text{ a partition of } [a, b] \right\}.$$

Since f is assumed to be bounded and the interval $[a, b]$ has finite length, the lower and upper Riemann integrals are finite. The upper integral is always at least as large as the lower integral, and if the two are equal we say that f is **Riemann integrable** over $[a, b]$[2] and call this common value the Riemann integral of f over $[a, b]$. We denote it by

$$(R)\int_a^b f$$

to temporarily distinguish it from the Lebesgue integral, which we consider in the next section.

A real-valued function ψ defined on $[a, b]$ is called a **step function** provided there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and numbers $c_1, \ldots, c_n$ such that for $1 \leq i \leq n$,

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i.$$

Observe that

$$L(\psi, P) = \sum_{i=1}^{n} c_i(x_i - x_{i-1}) = U(\psi, P).$$

---

[1]If we define

$$m_i = \inf \left\{ f(x) \mid x_{i-1} \leq x \leq x_i \right\} \text{ and } M_i = \sup \left\{ f(x) \mid x_{i-1} \leq x \leq x_i \right\},$$

so the infima and suprema are taken over closed subintervals, we arrive at the same value of the upper and lower Riemann integral.

[2]An elegant theorem of Henri Lebesgue, Theorem 8 of Chapter 5, tells us that a necessary and sufficient condition for a bounded function f to be Riemann integrable over $[a, b]$ is that the set of points in $[a, b]$ at which f fails to be continuous has Lebesgue measure zero.

From this and the definition of the upper and lower Riemann integrals, we infer that a step function ψ is Riemann integrable and

$$(R)\int_a^b \psi = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

Therefore, we may reformulate the definition of the lower and upper Riemann integrals as follows:

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \varphi \;\middle|\; \varphi \text{ a step function and } \varphi \leq f \text{ on } [a, b]\right\},$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \psi \;\middle|\; \psi \text{ a step function and } \psi \geq f \text{ on } [a, b]\right\}.$$

Example (Dirichlet's Function) Define f on $[0, 1]$ by setting $f(x) = 1$ if x is rational and 0 if x is irrational. Let P be any partition of $[0, 1]$. By the density of the rationals and the irrationals,[3]

$$L(f, P) = 0 \text{ and } U(f, P) = 1.$$

Thus

$$(R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f,$$

so f is not Riemann integrable. The set of rational numbers in [0,1] is countable.[4] Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[0, 1]$. For a natural number n, define f_n on $[0, 1]$ by setting $f_n(x) = 1$, if $x = q_k$ for some q_k with $1 \leq k \leq n$, and $f(x) = 0$ otherwise. Then each f_n is a step function, so it is Riemann integrable. Thus, $\{f_n\}$ is an increasing sequence of Riemann integrable functions on $[0, 1]$,

$$|f_n| \leq 1 \text{ on } [0, 1] \text{ for all } n$$

and

$$\{f_n\} \to f \text{ pointwise on } [0, 1].$$

However, the limit function f fails to be Riemann integrable on $[0, 1]$.

PROBLEMS

1. Show that, in the above Dirichlet function example, $\{f_n\}$ fails to converge to f uniformly on $[0, 1]$.
2. A partition P' of $[a, b]$ is called a refinement of a partition P provided each partition point of P is also a partition point of P'. For a bounded function f on $[a, b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

---

[3] See page 12.
[4] See page 14.

3. Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.
4. Suppose the bounded function f on $[a, b]$ is Riemann integrable over $[a, b]$. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ for which $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$.
5. Let f be a bounded function on $[a, b]$. Suppose there is a sequence $\{P_n\}$ of partitions of $[a, b]$ for which $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$. Show that f is Riemann integrable over $[a, b]$.
6. Use the preceding problem to show that since a continuous function f on a closed, bounded interval $[a, b]$ is uniformly continuous on $[a, b]$, it is Riemann integrable over $[a, b]$.
7. Let f be an increasing real-valued function on $[0, 1]$. For a natural number n, define P_n to be the partition of $[0, 1]$ into n subintervals of length $1/n$. Show that $U(f, P_n) - L(f, P_n) \leq 1/n[f(1) - f(0)]$. Use Problem 5 to show that f is Riemann integrable over $[0, 1]$.
8. Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to f on the closed, bounded interval $[a, b]$. If each f_n is Riemann integrable over $[a, b]$, show that f also is Riemann integrable over $[a, b]$. Is it true that

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f\,?$$

4.2 THE LEBESGUE INTEGRAL OF A BOUNDED MEASURABLE FUNCTION OVER A SET OF FINITE MEASURE

The Dirichlet function, which was examined in the preceding section, exhibits one of the principal shortcomings of the Riemann integral: a uniformly bounded sequence of Riemann integrable functions on a closed, bounded interval can converge pointwise to a function that is not Riemann integrable. We will see that the Lebesgue integral does not suffer from this shortcoming.

Henceforth we only consider the Lebesgue integral, unless explicitly mentioned otherwise, and so we use the pure integral symbol to denote the Lebesgue integral. The forthcoming Theorem 3 tells us that any bounded function that is Riemann integrable over $[a, b]$ is also Lebesgue integrable over $[a, b]$ and the two integrals are equal.

Recall that a measurable real-valued function ψ defined on a set E is said to be simple provided it takes only a finite number of real values. If ψ takes the distinct values $a_1, \ldots, a_n$ on E, then, by the measurability of ψ, its level sets $\psi^{-1}(a_i)$ are measurable and we have the canonical representation of ψ on E as

$$\psi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ where each } E_i = \psi^{-1}(a_i) = \{x \in E \mid \psi(x) = a_i\}. \tag{1}$$

The canonical representation is characterized by the E_i's being disjoint and the a_i's being distinct.

Definition *For a simple function ψ defined on a set of finite measure E, we define the integral of ψ over E by*

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot m(E_i),$$

where ψ has the canonical representation given by (1).

Lemma 1 *Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \le i \le n$, let a_i be a real number.*

$$\text{If } \varphi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ then } \int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Proof The collection $\{E_i\}_{i=1}^n$ is disjoint but the above may not be the canonical representation since the a_i's may not be distinct. We must account for possible repetitions. Let $\{\lambda_1, \ldots, \lambda_m\}$ be the distinct values taken by φ. For $1 \le j \le m$, set $A_j = \{x \in E \mid \varphi(x) = \lambda_j\}$. By definition of the integral in terms of canonical representations,

$$\int_E \varphi = \sum_{j=1}^{m} \lambda_j \cdot m(A_j).$$

For $1 \le j \le m$, let I_j be the set of indices i in $\{1, \ldots, n\}$ for which $a_i = \lambda_j$. Then $\{1, \ldots, n\} = \bigcup_{j=1}^m I_j$, and the union is disjoint. Moreover, by finite additivity of measure,

$$m(A_j) = \sum_{i \in I_j} m(E_i) \text{ for all } 1 \le j \le m.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} a_i \cdot m(E_i) &= \sum_{j=1}^{m} \left[\sum_{i \in I_j} a_i \cdot m(E_i) \right] = \sum_{j=1}^{m} \lambda_j \left[\sum_{i \in I_j} m(E_i) \right] \\ &= \sum_{j=1}^{m} \lambda_j \cdot m(A_j) = \int_E \varphi. \end{aligned}$$ □

One of our goals is to establish linearity and monotonicity properties for the general Lebesgue integral. The following is the first result in this direction.

Proposition 2 (Linearity and Monotonicity of Integration) *Let φ and ψ be simple functions defined on a set of finite measure E. Then for any α and β,*

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi.$$

Moreover,

$$\text{if } \varphi \le \psi \text{ on } E, \text{ then } \int_E \varphi \le \int_E \psi.$$

Proof Since both φ and ψ take only a finite number of values on E, we may choose a finite disjoint collection $\{E_i\}_{i=1}^n$ of measurable subsets of E, the union of which is E, such that φ and ψ are constant on each E_i. For each i, $1 \le i \le n$, let a_i and b_i, respectively, be the values taken by φ and ψ on E_i. By the preceding lemma,

$$\int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i) \text{ and } \int_E \psi = \sum_{i=1}^{n} b_i \cdot m(E_i)$$

However, the simple function $\alpha\varphi + \beta\psi$ takes the constant value $\alpha a_i + \beta b_i$ on E_i. Thus, again by the preceding lemma,

$$\begin{aligned}\int_E (\alpha\varphi + \beta\psi) &= \sum_{i=1}^{n} (\alpha a_i + \beta b_i) \cdot m(E_i) \\ &= \alpha \sum_{i=1}^{n} a_i \cdot m(E_i) + \beta \sum_{i=1}^{n} b_i \cdot m(E_i) = \alpha \int_E \varphi + \beta \int_E \psi.\end{aligned}$$

To prove monotonicity, assume $\varphi \leq \psi$ on E. Define $\eta = \psi - \varphi$ on E. By linearity,

$$\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E \eta \geq 0,$$

since the nonnegative simple function η has a nonnegative integral. □

The linearity of integration over sets of finite measure of simple functions shows that the restriction in the statement of Lemma 1 that the collection $\{E_i\}_{i=1}^n$ be disjoint is unnecessary.

A step function takes only a finite number of values and each interval is measurable. Thus a step function is simple. Since the measure of a singleton set is zero and the measure of an interval is its length, we infer from the linearity of Lebesgue integration for simple functions defined on sets of finite measure that the Riemann integral over a closed, bounded interval of a step function agrees with the Lebesgue integral.

Let f be a bounded real-valued function defined on a set of finite measure E. By analogy with the Riemann integral, we define the **lower and upper Lebesgue integral**, respectively, of f over E to be

$$\sup \left\{ \int_E \varphi \;\middle|\; \varphi \text{ simple and } \varphi \leq f \text{ on } E, \right\}$$

and

$$\inf \left\{ \int_E \psi \;\middle|\; \psi \text{ simple and } f \leq \psi \text{ on } E. \right\}$$

Since f is assumed to be bounded, by the monotonicity property of the integral for simple functions, the lower and upper integrals are finite and the upper integral is always at least as large as the lower integral.

Definition *A bounded function f on a domain E of finite measure is said to be* **Lebesgue integrable** *over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the* **Lebesgue integral**, *or simply the integral, of f over E and is denoted by $\int_E f$.*

Theorem 3 *Let f be a bounded function defined on the closed, bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.*

Proof The assertion that f is Riemann integrable means that, setting $I = [a, b]$,

$$\sup\left\{(R)\int_I \varphi \;\middle|\; \varphi \text{ a step function}, \varphi \le f\right\} = \inf\left\{(R)\int_I \psi \;\middle|\; \psi \text{ a step function}, f \le \psi\right\}$$

To prove that f is Lebesgue integrable we must show that

$$\sup\left\{\int_I \varphi \;\middle|\; \varphi \text{ simple}, \varphi \le f\right\} = \inf\left\{\int_I \psi \;\middle|\; \psi \text{ simple}, f \le \psi\right\}.$$

However, each step function is a simple function and, as we have already observed, for a step function, the Riemann integral and the Lebesgue integral are the same. Therefore the first equality implies the second and also the equality of the Riemann and Lebesgue integrals. □

We are now fully justified in using the symbol $\int_E f$, without any preliminary (R), to denote the integral of a bounded function that is Lebesgue integrable over a set of finite measure. In the case of an interval $E = [a, b]$, we sometimes use the familiar notation $\int_a^b f$ to denote $\int_{[a, b]} f$ and sometimes it is useful to use the classic Leibniz notation $\int_a^b f(x)\,dx$.

Example The set E of rational numbers in $[0, 1]$ is a measurable set of measure zero. The Dirichlet function f is the restriction to $[0, 1]$ of the characteristic function of E, χ_E. Thus f is integrable over $[0, 1]$ and

$$\int_{[0,1]} f = \int_{[0,1]} 1 \cdot \chi_E = 1 \cdot m(E) = 0.$$

We have shown that f is not Riemann integrable over $[0, 1]$.

Theorem 4 *Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.*

Proof Let n be a natural number. By the Simple Approximation Lemma, with $\epsilon = 1/n$, there are two simple functions φ_n and ψ_n defined on E for which

$$\varphi_n \le f \le \psi_n \text{ on } E,$$

and

$$0 \le \psi_n - \varphi_n \le 1/n \text{ on } E.$$

By the monotonicity and linearity of the integral for simple functions,

$$0 \le \int_E \psi_n - \int_E \varphi_n = \int_E [\psi_n - \varphi_n] \le 1/n \cdot m(E).$$

However,

$$0 \leq \inf\left\{\int_E \psi \;\middle|\; \psi \text{ simple}, \psi \geq f\right\} - \sup\left\{\int_E \varphi \;\middle|\; \varphi \text{ simple}, \varphi \leq f\right\}$$
$$\leq \int_E \psi_n - \int_E \varphi_n \leq 1/n \cdot m(E).$$

This inequality holds for every natural number n and $m(E)$ is finite. Therefore the upper and lower Lebesgue integrals are equal and thus the function f is integrable over E. □

It turns out that the converse of the preceding theorem is true; a bounded function on a set of finite measure is Lebesgue integrable if and only if it is measurable: we prove this later (see the forthcoming Theorem 7 of Chapter 5). This shows, in particular, that not every bounded function defined on a set of finite measure is Lebesgue integrable. In fact, for any measurable set E of finite positive measure, the restriction to E of the characteristic function of each nonmeasurable subset of E fails to be Lebesgue integrable over E.

Theorem 5 (Linearity and Monotonicity of Integration) *Let f and g be bounded measurable functions on a set of finite measure E. Then for any α and β,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \tag{2}$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g. \tag{3}$$

Proof A linear combination of measurable bounded functions is measurable and bounded. Thus, by Theorem 4, $\alpha f + \beta g$ is integrable over E. We first prove linearity for $\beta = 0$. If ψ is a simple function so is $\alpha\psi$, and conversely (if $\alpha \neq 0$). We established linearity of integration for simple functions. Let $\alpha > 0$. Since the Lebesgue integral is equal to the upper Lebesgue integral,

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \int_E \psi = \alpha \inf_{[\psi/\alpha] \geq f} \int_E [\psi/\alpha] = \alpha \int_E f.$$

For $\alpha < 0$, since the Lebesgue integral is equal both to the upper Lebesgue integral and the lower Lebesgue integral,

$$\int_E \alpha f = \inf_{\varphi \geq \alpha f} \int_E \varphi = \alpha \sup_{[\varphi/\alpha] \leq f} \int_E [\varphi/\alpha] = \alpha \int_E f.$$

It remains to establish linearity in the case that $\alpha = \beta = 1$. Let ψ_1 and ψ_2 be simple functions for which $f \leq \psi_1$ and $g \leq \psi_2$ on E. Then $\psi_1 + \psi_2$ is a simple function and $f + g \leq \psi_1 + \psi_2$ on E. Hence, since $\int_E (f + g)$ is equal to the upper Lebesgue integral of $f + g$ over E, by the linearity of integration for simple functions,

$$\int_E (f + g) \leq \int_E (\psi_1 + \psi_2) = \int_E \psi_1 + \int_E \psi_2.$$

The greatest lower bound for the sums of integrals on the right-hand side, as ψ_1 and ψ_2 vary among simple functions for which $f \leq \psi_1$ and $g \leq \psi_2$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is a lower bound for these same sums. Therefore,

$$\int_E (f+g) \leq \int_E f + \int_E g.$$

It remains to prove this inequality in the opposite direction. Let φ_1 and φ_2 be simple functions for which $\varphi_1 \leq f$ and $\varphi_2 \leq g$ on E. Then $\varphi_1 + \varphi_2 \leq f + g$ on E and $\varphi_1 + \varphi_2$ is simple. Hence, since $\int_E (f+g)$ is equal to the lower Lebesgue integral of $f+g$ over E, by the linearity of integration for simple functions,

$$\int_E (f+g) \geq \int_E (\varphi_1 + \varphi_2) = \int_E \varphi_1 + \int_E \varphi_2.$$

The least upper bound bound for the sums of integrals on the right-hand side, as φ_1 and φ_2 vary among simple functions for which $\varphi_1 \leq f$ and $\varphi_2 \leq g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is an upper bound for these same sums. Therefore,

$$\int_E (f+g) \geq \int_E f + \int_E g.$$

This completes the proof of linearity of integration.

To prove monotonicity, assume $f \leq g$ on E. Define $h = g - f$ on E. By linearity,

$$\int_E g - \int_E f = \int_E (g - f) = \int_E h.$$

The function h is nonnegative and therefore $\psi \leq h$ on E, where $\psi \equiv 0$ on E. Since the integral of h equals its lower integral, $\int_E h \geq \int_E \psi = 0$. Therefore, $\int_E f \leq \int_E g$. □

Corollary 6 *Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then*

$$\int_{A \cup B} f = \int_A f + \int_B f. \tag{4}$$

Proof Both $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable functions on E. Since A and B are disjoint,

$$f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B.$$

Furthermore, for any measurable subset E_1 of E (see Problem 10),

$$\int_{E_1} f = \int_E f \cdot \chi_{E_1}.$$

Therefore, by the linearity of integration,

$$\int_{A \cup B} f = \int_E f \cdot \chi_{A \cup B} = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B = \int_A f + \int_B f.$$ □

Corollary 7 *Let f be a bounded measurable function on a set of finite measure E. Then*

$$\left|\int_E f\right| \leq \int_E |f|. \tag{5}$$

Proof The function $|f|$ is measurable and bounded. Now

$$-|f| \leq f \leq |f| \text{ on } E.$$

By the linearity and monotonicity of integration,

$$-\int_E |f| \leq \int_E f \leq \int_E |f|,$$

that is, (5) holds. □

Proposition 8 *Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E.*

$$\textit{If } \{f_n\} \to f \textit{ uniformly on } E, \textit{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof Since the convergence is uniform and each f_n is bounded, the limit function f is bounded. The function f is measurable since it is the pointwise limit of a sequence of measurable functions. Let $\epsilon > 0$. Choose an index N for which

$$|f - f_n| < \epsilon/m(E) \text{ on } E \text{ for all } n \geq N. \tag{6}$$

By the linearity and monotonicity of integration and the preceding corollary, for each $n \geq N$,

$$\left|\int_E f - \int_E f_n\right| = \left|\int_E [f - f_n]\right| \leq \int_E |f - f_n| \leq [\epsilon/m(E)] \cdot m(E) = \epsilon.$$

Therefore $\lim_{n\to\infty} \int_E f_n = \int_E f$. □

This proposition is rather weak since frequently a sequence will be presented that converges pointwise but not uniformly. It is important to understand when it is possible to infer from

$$\{f_n\} \to f \text{ pointwise a.e. on } E$$

that

$$\lim_{n\to\infty}\left[\int_E f_n\right] = \int_E \left[\lim_{n\to\infty} f\right] = \int_E f.$$

We refer to this equality as **passage of the limit under the integral sign.**[5] Before proving our first important result regarding this passage, we present an instructive example.

[5]This phrase is taken from I. P. Natanson's *Theory of Functions of a Real Variable* [Nat55].

Example For each natural number n, define f_n on $[0, 1]$ to have the value 0 if $x \geq 2/n$, have $f(1/n) = n$, $f(0) = 0$ and to be linear on the intervals $[0, 1/n]$ and $[1/n, 2/n]$. Observe that $\int_0^1 f_n = 1$ for each n. Define $f \equiv 0$ on $[0, 1]$. Then

$$\{f_n\} \to f \text{ pointwise on } [0, 1], \text{ but } \lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f.$$

Thus, pointwise convergence alone is not sufficient to justify passage of the limit under the integral sign.

The Bounded Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which*

$$|f_n| \leq M \text{ on } E \text{ for all } n.$$

$$\text{If } \{f_n\} \to f \text{ pointwise on } E, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof The proof of this theorem furnishes a nice illustration of Littlewood's Third Principle. If the convergence is uniform, we have the easy proof of the preceding proposition. However, Egoroff's Theorem tells us, roughly, that pointwise convergence is "nearly" uniform.

The pointwise limit of a sequence of measurable functions is measurable. Therefore f is measurable. Clearly $|f| \leq M$ on E. Let A be any measurable subset of E and n a natural number. By the linearity and additivity over domains of the integral,

$$\int_E f_n - \int_E f = \int_E [f_n - f] = \int_A [f_n - f] + \int_{E\sim A} f_n + \int_{E\sim A} (-f).$$

Therefore, by Corollary 7 and the monotonicity of integration,

$$\left| \int_E f_n - \int_E f \right| \leq \int_A |f_n - f| + 2M \cdot m(E \sim A). \tag{7}$$

To prove convergence of the integrals, let $\epsilon > 0$. Since $m(E) < \infty$ and f is real-valued, Egoroff's Theorem tells us that there is a measurable subset A of E for which $\{f_n\} \to f$ uniformly on A and $m(E \sim A) < \epsilon/4M$. By uniform convergence, there is an index N for which

$$|f_n - f| < \frac{\epsilon}{2 \cdot m(E)} \text{ on } A \text{ for all } n \geq N.$$

Therefore, for $n \geq N$, we infer from (7) and the monotonicity of integration that

$$\left| \int_E f_n - \int_E f \right| \leq \frac{\epsilon}{2 \cdot m(E)} \cdot m(A) + 2M \cdot m(E \sim A) < \epsilon.$$

Hence the sequence of integrals $\{\int_E f_n\}$ converges to $\int_E f$. □

Remark *Prior to the proof of the Bounded Convergence Theorem, no use was made of the countable additivity of Lebesgue measure on the real line. Only finite additivity was used, and it was used just once, in the proof of Lemma 1. But for the proof of the Bounded Convergence Theorem we used Egoroff's Theorem. The proof of Egoroff's Theorem needed the continuity of Lebesgue measure, a consequence of countable additivity of Lebesgue measure.*

PROBLEMS

9. Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.
10. Let f be a bounded measurable function on a set of finite measure E. For a measurable subset A of E, show that $\int_A f = \int_E f \cdot \chi_A$.
11. Does the Bounded Convergence Theorem hold for the Riemann integral?
12. Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. Show that $\int_E f = \int_E g$.
13. Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on E?
14. Show that Proposition 8 is a special case of the Bounded Convergence Theorem.
15. Verify the assertions in the last Remark of this section.
16. Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on E.

4.3 THE LEBESGUE INTEGRAL OF A MEASURABLE NONNEGATIVE FUNCTION

A measurable function f on E is said to vanish outside a set of finite measure provided there is a subset E_0 of E for which $m(E_0) < \infty$ and $f \equiv 0$ on $E \sim E_0$. It is convenient to say that a function that vanishes outside a set of finite measure has finite support and define its support to be $\{x \in E \mid f(x) \neq 0\}$.[6] In the preceding section, we defined the integral of a bounded measurable function f over a set of finite measure E. However, even if $m(E) = \infty$, if f is bounded and measurable on E but has finite support, we can define its integral over E by

$$\int_E f = \int_{E_0} f,$$

where E_0 has finite measure and $f \equiv 0$ on $E \sim E_0$. This integral is properly defined, that is, it is independent of the choice of set of finite measure E_0 outside of which f vanishes. This is a consequence of the additivity over domains property of integration for bounded measurable functions over a set of finite measure.

Definition *For f a nonnegative measurable function on E, we define the integral of f over E by*[7]

$$\int_E f = \sup\left\{\int_E h \;\middle|\; h \text{ bounded, measurable, of finite support and } 0 \le h \le f \text{ on } E\right\}. \tag{8}$$

[6] But care is needed here. In the study of continuous real-valued functions on a topological space, the support of a function is defined to be the closure of the set of points at which the function is nonzero.

[7] This is a definition of the integral of a nonnegative extended real-valued measurable function; it is not a definition of what it means for such a function to be integrable. The integral is defined regardless of whether the function is bounded or the domain has finite measure. Of course, the integral is nonnegative since it is defined to be the supremum of a set of nonnegative numbers. But the integral may be equal to ∞, as it is, for instance, for a nonnegative measurable function that takes a positive constant value of a subset of E of infinite measure or the value ∞ on a subset of E of positive measure.

Chebychev's Inequality *Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,*

$$m\left\{x \in E \mid f(x) \geq \lambda\right\} \leq \frac{1}{\lambda} \cdot \int_E f. \tag{9}$$

Proof Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$. First suppose $m(E_\lambda) = \infty$. Let n be a natural number. Define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \cdot \chi_{E_{\lambda,n}}$. Then ψ_n is a bounded measurable function of finite support,

$$\lambda \cdot m(E_{\lambda,n}) = \int_E \psi_n \text{ and } 0 \leq \psi_n \leq f \text{ on } E \text{ for all } n.$$

We infer from the continuity of measure that

$$\infty = \lambda \cdot m(E_\lambda) = \lambda \cdot \lim_{n \to \infty} m(E_{\lambda,n}) = \lim_{n \to \infty} \int_E \psi_n \leq \int_E f.$$

Thus inequality (9) holds since both sides equal ∞. Now consider the case $m(E_\lambda) < \infty$. Define $h = \lambda \cdot \chi_{E_\lambda}$. Then h is a bounded measurable function of finite support and $0 \leq h \leq f$ on E. By the definition of the integral of f over E,

$$\lambda \cdot m(E_\lambda) = \int_E h \leq \int_E f.$$

Divide both sides of this inequality by λ to obtain Chebychev's Inequality. □

Proposition 9 *Let f be a nonnegative measurable function on E. Then*

$$\int_E f = 0 \textit{ if and only if } f = 0 \textit{ a.e. on } E. \tag{10}$$

Proof First assume $\int_E f = 0$. Then, by Chebychev's Inequality, for each natural number n, $m\{x \in X \mid f(x) \geq 1/n\} = 0$. By the countable additivity of Lebesgue measure, $m\{x \in X \mid f(x) > 0\} = 0$. Conversely, suppose $f = 0$ a.e. on E. Let φ be a simple function and h a bounded measurable function of finite support for which $0 \leq \varphi \leq h \leq f$ on E. Then $\varphi = 0$ a.e. on E and hence $\int_E \varphi = 0$. Since this holds for all such φ, we infer that $\int_E h = 0$. Since this holds for all such h, we infer that $\int_E f = 0$. □

Theorem 10 (Linearity and Monotonicity of Integration) *Let f and g be nonnegative measurable functions on E. Then for any $\alpha > 0$ and $\beta > 0$,*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g. \tag{11}$$

Moreover,

$$\textit{if } f \leq g \textit{ on } E, \textit{ then } \int_E f \leq \int_E g. \tag{12}$$

Proof For $\alpha > 0$, $0 \le h \le f$ on E if and only if $0 \le \alpha h \le \alpha f$ on E. Therefore, by the linearity of the integral of bounded functions of finite support, $\int_E \alpha f = \alpha \int_E f$. Thus, to prove linearity we need only consider the case $\alpha = \beta = 1$. Let h and g be bounded measurable functions of finite support for which $0 \le h \le f$ and $0 \le k \le g$ on E. We have $0 \le h + k \le f + g$ on E, and $h + k$ also is a bounded measurable function of finite support. Thus, by the linearity of integration for bounded measurable functions of finite support,

$$\int_E h + \int_E k = \int_E (h+k) \le \int_E (f+g).$$

The least upper bound for the sums of integrals on the left-hand side, as h and k vary among bounded measurable functions of finite support for which $h \le f$ and $k \le g$, equals $\int_E f + \int_E g$. These inequalities tell us that $\int_E (f+g)$ is an upper bound for these same sums. Therefore,

$$\int_E f + \int_E g \le \int_E (f+g).$$

It remains to prove this inequality in the opposite direction, that is,

$$\int_E (f+g) \le \int_E f + \int_E g.$$

By the definition of $\int_E (f+g)$ as the supremum of $\int_E \ell$ as ℓ ranges over all bounded measurable functions of finite support for which $0 \le \ell \le f + g$ on E, to verify this inequality it is necessary and sufficient to show that for any such function ℓ,

$$\int_E \ell \le \int_E f + \int_E g. \tag{13}$$

For such a function ℓ, define the functions h and k on E by

$$h = \min\{f, \ell\} \text{ and } k = \ell - h \text{ on } E.$$

Let x belong to E. If $\ell(x) \le f(x)$, then $k(x) = 0 \le g(x)$; if $\ell(x) > f(x)$, then $h(x) = \ell(x) - f(x) \le g(x)$. Therefore, $h \le g$ on E. Both h and k are bounded measurable functions of finite support. We have

$$0 \le h \le f, 0 \le k \le g \text{ and } \ell = h + k \text{ on } E.$$

Hence, again using the linearity of integration for bounded measurable functions of finite support and the definitions of $\int_E f$ and $\int_E g$, we have

$$\int_E l = \int_E h + \int_E k \le \int_E f + \int_E g.$$

Thus (13) holds and the proof of linearity is complete.

In view of the definition of $\int_E f$ as a supremum, to prove the monotonicity inequality (12) it is necessary and sufficient to show that if h is a bounded measurable function of finite support for which $0 \le h \le f$ on E, then

$$\int_E h \le \int_E g. \tag{14}$$

Let h be such a function. Then $h \leq g$ on E. Therefore, by the definition of $\int_E g$ as a supremum, $\int_E h \leq \int_E g$. This completes the proof of monotonicity. □

Theorem 11 (Additivity Over Domains of Integration) *Let f be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E, then*

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

In particular, if E_0 is a subset of E of measure zero, then

$$\int_E f = \int_{E \sim E_0} f. \tag{15}$$

Proof Additivity over domains of integration follows from linearity as it did for bounded functions on sets of finite measure. The excision formula (15) follows from additivity over domains and the observation that, by Proposition 9, the integral of a nonnegative function over a set of measure zero is zero. □

The following lemma will enable us to establish several criteria to justify passage of the limit under the integral sign.

Fatou's Lemma *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\textit{If } \{f_n\} \rightarrow f \textit{ pointwise a.e. on } E, \textit{ then } \int_E f \leq \liminf \int_E f_n. \tag{16}$$

Proof In view of (15), by possibly excising from E a set of measure zero, we assume the pointwise convergence is on all of E. The function f is nonnegative and measurable since it is the pointwise limit of a sequence of such functions. To verify the inequality in (16) it is necessary and sufficient to show that if h is any bounded measurable function of finite support for which $0 \leq h \leq f$ on E, then

$$\int_E h \leq \liminf \int_E f_n. \tag{17}$$

Let h be such a function. Choose $M \geq 0$ for which $|h| \leq M$ on E. Define $E_0 = \{x \in E \mid h(x) \neq 0\}$. Then $m(E_0) < \infty$. Let n be a natural number. Define a function h_n on E by

$$h_n = \min\{h, f_n\} \text{ on } E.$$

Observe that the function h_n is measurable, that

$$0 \leq h_n \leq M \text{ on } E_0 \text{ and } h_n \equiv 0 \text{ on } E \sim E_0.$$

Furthermore, for each x in E, since $h(x) \leq f(x)$ and $\{f_n(x)\} \rightarrow f(x)$, $\{h_n(x)\} \rightarrow h(x)$. We infer from the Bounded Convergence Theorem applied to the uniformly bounded sequence of restrictions of h_n to the set of finite measure E_0, and the vanishing of each h_n on $E \sim E_0$, that

$$\lim_{n \rightarrow \infty} \int_E h_n = \lim_{n \rightarrow \infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h.$$

However, for each n, $h_n \leq f_n$ on E and therefore, by the definition of the integral of f_n over E, $\int_E h_n \leq \int_E f_n$. Thus,

$$\int_E h = \lim_{n\to\infty} \int_E h_n \leq \liminf \int_E f_n. \qquad \square$$

The inequality in Fatou's Lemma may be strict.

Example Let $E = (0, 1]$ and for a natural number n, define $f_n = n \cdot \chi_{(0, 1/n)}$. Then $\{f_n\}$ converges pointwise on E to $f \equiv 0$ on E. However,

$$\int_E f = 0 < 1 = \lim_{n\to\infty} \int_E f_n.$$

As another example of strict inequality in Fatou's Lemma, let $E = \mathbf{R}$ and for a natural number n, define $g_n = \chi_{(n, n+1)}$. Then $\{g_n\}$ converges pointwise on E to $g \equiv 0$ on E. However,

$$\int_E g = 0 < 1 = \lim_{n\to\infty} \int_E g_n.$$

However, the inequality in Fatou's Lemma is an equality if the sequence $\{f_n\}$ is increasing.

The Monotone Convergence Theorem *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E.*

$$\textit{If } \{f_n\} \to f \textit{ pointwise a.e. on } E, \textit{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof According to Fatou's Lemma,

$$\int_E f \leq \liminf \int_E f_n.$$

However, for each index n, $f_n \leq f$ a.e. on E, and so, by the monotonicity of integration for nonnegative measurable functions and (15), $\int_E f_n \leq \int_E f$. Therefore

$$\limsup \int_E f_n \leq \int_E f.$$

Hence

$$\int_E f = \lim_{n\to\infty} \int f_n. \qquad \square$$

Corollary 12 *Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\textit{If } f = \sum_{n=1}^{\infty} u_n \textit{ pointwise a.e. on } E, \textit{ then } \int_E f = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof Apply the Monotone Convergence Theorem with $f_n = \sum_{k=1}^{n} u_k$, for each index n, and then use the linearity of integration for nonnegative measurable functions. $\square$

Definition *A nonnegative measurable function f on a measurable set E is said to be* **integrable** *over E provided*

$$\int_E f < \infty.$$

Proposition 13 *Let the nonnegative function f be integrable over E. Then f is finite a.e. on E.*

Proof Let n be a natural number. Chebychev's Inequality and the monotonicity of measure tell us that

$$m\left\{x \in E \mid f(x) = \infty\right\} \leq m\left\{x \in E \mid f(x) \geq n\right\} \leq \frac{1}{n}\int_E f.$$

But $\int_E f$ is finite and therefore $m\{x \in E \mid f(x) = \infty\} = 0$. □

Beppo Levi's Lemma *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e. on E and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f < \infty.$$

Proof Every monotone sequence of extended real numbers converges to an extended real number.[8] Since $\{f_n\}$ is an increasing sequence of extended real-valued functions on E, we may define the extended real-valued nonnegative function f pointwise on E by

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for all } x \in E.$$

According to the Monotone Convergence Theorem, $\{\int_E f_n\} \to \int_E f$. Therefore, since the sequence of real numbers $\{\int_E f_n\}$ is bounded, its limit is finite and so $\int_E f < \infty$. We infer from the preceding proposition that f is finite a.e. on E. □

PROBLEMS

17. Let E be a set of measure zero and define $f \equiv \infty$ on E. Show that $\int_E f = 0$.
18. Show that the integral of a bounded measurable function of finite support is properly defined.
19. For a number α, define $f(x) = x^\alpha$ for $0 < x \leq 1$, and $f(0) = 0$. Compute $\int_0^1 f$.
20. Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f pointwise on E. Let $M \geq 0$ be such that $\int_E f_n \leq M$ for all n. Show that $\int_E f \leq M$. Verify that this property is equivalent to the statement of Fatou's Lemma.
21. Let the function f be nonnegative and integrable over E and $\epsilon > 0$. Show there is a simple function η on E that has finite support, $0 \leq \eta \leq f$ on E and $\int_E |f - \eta| < \epsilon$. If E is a closed, bounded interval, show there is a step function h on E that has finite support and $\int_E |f - h| < \epsilon$.
22. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbf{R}$ that converges pointwise on $\mathbf{R}$ to f and f be integrable over $\mathbf{R}$. Show that

$$\text{if } \int_{\mathbf{R}} f = \lim_{n\to\infty} \int_{\mathbf{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

[8] See page 23.

23. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n+1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.
24. Let f be a nonnegative measurable function on E.
    (i) Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on E, each of finite support, which converges pointwise on E to f.
    (ii) Show that $\int_E f = \sup \{\int_E \varphi \mid \varphi$ simple, of finite support and $0 \leq \varphi \leq, f$ on $E\}$.
25. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \leq f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

26. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.
27. Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

4.4 THE GENERAL LEBESGUE INTEGRAL

For an extended real-valued function f on E, we have defined the positive part f^+ and the negative part f^- of f, respectively, by

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\} \text{ for all } x \in E.$$

Then f^+ and f^- are nonnegative functions on E,

$$f = f^+ - f^- \text{ on } E$$

and

$$|f| = f^+ + f^- \text{ on } E.$$

Observe that f is measurable if and only if both f^+ and f^- are measurable.

Proposition 14 *Let f be a measurable function on E. Then $f^+$and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof Assume f^+ and f^- are integrable nonnegative functions. By the linearity of integration for nonnegative functions, $|f| = f^+ + f^-$ is integrable over E. Conversely, suppose $|f|$ is integrable over E. Since $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$ on E, we infer from the monotonicity of integration for nonnegative functions that both f^+ and f^- are integrable over E. □

Definition *A measurable function f on E is said to be* **integrable** *over E provided $|f|$ is integrable over E. When this is so we define the integral of f over E by*

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Of course, for a nonnegative function f, since $f = f^+$ and $f^- \equiv 0$ on E, this definition of integral coincides with the one just considered. By the linearity of integration for bounded measurable functions of finite support, the above definition of integral also agrees with the definition of integral for this class of functions.

Proposition 15 *Let f be integrable over E. Then f is finite a.e. on E and*

$$\int_E f = \int_{E \sim E_0} f \text{ if } E_0 \subseteq E \text{ and } m(E_0) = 0. \tag{18}$$

Proof Proposition 13, tells us that $|f|$ is finite a.e. on E. Thus f is finite a.e. on E. Moreover, (18) follows by applying (15) to the positive and negative parts of f. □

The following criterion for integrability is the Lebesgue integral correspondent of the comparison test for the convergence of series of real numbers.

Proposition 16 (the Integral Comparison Test) *Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and dominates f in the sense that*

$$|f| \leq g \text{ on } E.$$

Then f is integrable over E and

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof By the monotonicity of integration for nonnegative functions, $|f|$, and hence f, is integrable. By the triangle inequality for real numbers and the linearity of integration for nonnegative functions,

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right| \leq \int_E f^+ + \int_E f^- = \int_E |f|.$$

□

We have arrived at our final stage of generality for the Lebesgue integral for functions of a single real variable. Before proving the linearity property for integration, we need to address, with respect to integration, a point already addressed with respect to measurability. The point is that for two functions f and g which are integrable over E, the sum $f + g$ is not properly defined at points in E where f and g take infinite values of opposite sign. However, by Proposition 15, if we define A to be the set of points in E at which both f and g are finite, then $m(E \sim A) = 0$. Once we show that $f + g$ is integrable over A, we define

$$\int_E (f + g) = \int_A (f + g).$$

We infer from (18) that $\int_E (f + g)$ is equal to the integral over E of any extension of $(f + g)|_A$ to an extended real-valued function on all of E.

Theorem 17 (Linearity and Monotonicity of Integration) *Let the functions f and g be integrable over E. Then for any α and β, the function $\alpha f + \beta g$ is integrable over E and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g.$$

Moreover,

$$\text{if } f \leq g \text{ on } E, \text{ then } \int_E f \leq \int_E g.$$

Proof If $\alpha > 0$, then $[\alpha f]^+ = \alpha f^+$ and $[\alpha f]^- = \alpha f^-$, while if $\alpha < 0$, $[\alpha f]^+ = -\alpha f^-$ and $[\alpha f]^- = -\alpha f^+$. Therefore $\int_E \alpha f = \alpha \int_E f$, since we established this for nonnegative functions f and $\alpha > 0$. So it suffices to establish linearity in the case $\alpha = \beta = 1$. By the linearity of integration for nonnegative functions, $|f| + |g|$ is integrable over E. Since $|f + g| \leq |f| + |g|$ on E, by the integral comparison test, $f + g$ also is integrable over E. Proposition 15 tells us that f and g are finite a.e. on E. According to the same proposition, by possibly excising from E a set of measure zero, we may assume that f and g are finite on E. To verify linearity is to show that

$$\int_E [f+g]^+ - \int_E [f+g]^- = \left[\int_E f^+ - \int_E f^-\right] + \left[\int_E g^+ - \int_E g^-\right]. \tag{19}$$

But

$$(f+g)^+ - (f+g)^- = f + g = (f^+ - f^-) + (g^+ - g^-) \text{ on } E,$$

and therefore, since each of these six functions takes real values on E,

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+ \text{ on } E.$$

We infer from linearity of integration for nonnegative functions that

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+.$$

Since f, g and $f + g$ are integrable over E, each of these six integrals is finite. Rearrange these integrals to obtain (19). This completes the proof of linearity.

To establish monotonicity we again argue as above that we may assume g and f are finite on E. Define $h = g - f$ on E. Then h is a properly defined nonnegative measurable function on E. By linearity of integration for integrable functions and monotonicity of integration for nonnegative functions,

$$\int_E g - \int_E f = \int_E (g - f) = \int_E h \geq 0. \qquad \square$$

Corollary 18 (Additivity Over Domains of Integration) *Let f be integrable over E. Assume A and B are disjoint measurable subsets of E. Then*

$$\int_{A \cup B} f = \int_A f + \int_B f. \tag{20}$$

Proof Observe that $|f \cdot \chi_A| \le |f|$ and $|f \cdot \chi_B| \le |f|$ on E. By the integral comparison test, the measurable functions $f \cdot \chi_A$ and $f \cdot \chi_B$ are integrable over E. Since A and B are disjoint

$$f \cdot \chi_{A \cup B} = f \cdot \chi_A + f \cdot \chi_B \text{ on } E. \tag{21}$$

But for any measurable subset C of E (see Problem 28),

$$\int_C f = \int_E f \cdot \chi_C.$$

Thus (20) follows from (21) and the linearity of integration. □

The following generalization of the Bounded Convergence Theorem provides another justification for passage of the limit under the integral sign.

The Lebesgue Dominated Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \le g$ on E for all n.*

If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f.$

Proof Since $|f_n| \le g$ on E and $|f| \le g$ a.e. on E and g is integrable over E, by the integral comparison test, f and each f_n also are integrable over E. We infer from Proposition 15 that, by possibly excising from E a countable collection of sets of measure zero and using the countable additivity of Lebesgue measure, we may assume that f and each f_n is finite on E. The function $g - f$ and for each n, the function $g - f_n$, are properly defined, nonnegative and measurable. Moreover, the sequence $\{g - f_n\}$ converges pointwise a.e. on E to $g - f$. Fatou's Lemma tells us that

$$\int_E (g - f) \le \liminf \int_E (g - f_n).$$

Thus, by the linearity of integration for integrable functions,

$$\int_E g - \int_E f = \int_E (g - f) \le \liminf \int_E (g - f_n) = \int_E g - \limsup \int_E f_n,$$

that is,

$$\limsup \int_E f_n \le \int_E f.$$

Similarly, considering the sequence $\{g + f_n\}$, we obtain

$$\int_E f \le \liminf \int_E f_n.$$

The proof is complete. □

The following generalization of the Lebesgue Dominated Convergence Theorem, the proof of which we leave as an exercise (see Problem 32), is often useful (see Problem 33).

Theorem 19 (General Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

$$\text{If } \lim_{n\to\infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Remark *In Fatou's Lemma and the Lebesgue Dominated Convergence Theorem, the assumption of pointwise convergence a.e. on E rather than on all of E is not a decoration pinned on to honor generality. It is necessary for future applications of these results. We provide one illustration of this necessity. Suppose f is an increasing function on all of* **R**. *A forthcoming theorem of Lebesgue (Lebesgue's Theorem of Chapter 6) tells us that*

$$\lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n} = f'(x) \text{ for almost all } x. \tag{22}$$

From this and Fatou's Lemma we will show that for any closed, bounded interval $[a, b]$,

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

In general, given a nondegenerate closed, bounded interval $[a, b]$ and a subset A of $[a, b]$ that has measure zero, there is an increasing function f on $[a, b]$ for which the limit in (22) fails to exist at each point in A (see Problem 10 of Chapter 6).

PROBLEMS

28. Let f be integrable over E and C a measurable subset of E. Show that $\int_C f = \int_E f \cdot \chi_C$.

29. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

30. Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

31. Let f be a measurable function on E which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative on E. Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f.