32. Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.
33. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)
34. Let f be a nonnegative measurable function on $\mathbf{R}$. Show that
$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbf{R}} f.$$
35. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and is a measurable function of x for each fixed value of y. Suppose for each fixed value of x, $\lim_{y\to 0} f(x, y) = f(x)$ and that for all y, we have $|f(x, y)| \le g(x)$, where g is integrable over $[0, 1]$. Show that
$$\lim_{y\to 0} \int_0^1 f(x, y)dx = \int_0^1 f(x)dx.$$
Also show that if the function $f(x, y)$ is continuous in y for each x, then
$$h(y) = \int_0^1 f(x, y)dx$$
is a continuous function of y.
36. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$ let the partial derivative $\partial f/\partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that
$$\left|\frac{\partial f}{\partial y}(x, y)\right| \le g(x) \text{ for all } (x, y) \in Q.$$
Prove that
$$\frac{d}{dy}\left[\int_0^1 f(x, y)\,dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x, y)\,dx \text{ for all } y \in [0, 1].$$

4.5 COUNTABLE ADDITIVITY AND CONTINUITY OF INTEGRATION

The linearity and monotonicity properties of the Lebesgue integral, which we established in the preceding section, are extensions of familiar properties of the Riemann integral. In this brief section we establish two properties of the Lebesgue integral which have no counterpart for the Riemann integral. The following countable additivity property for Lebesgue integration is a companion of the countable additivity property for Lebesgue measure.

Theorem 20 (the Countable Additivity of Integration) *Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E whose union is E. Then*
$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f. \tag{23}$$

Proof Let n be a natural number. Define $f_n = f \cdot \chi_n$ where χ_n is the characteristic function of the measurable set $\bigcup_{k=1}^n E_k$. Then f_n is a measurable function on E and

$$|f_n| \le |f| \text{ on } E.$$

Observe that $\{f_n\} \to f$ pointwise on E. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

On the other hand, since $\{E_n\}_{n=1}^\infty$ is disjoint, it follows from the additivity over domains property of the integral that for each n,

$$\int_E f_n = \sum_{k=1}^n \int_{E_k} f.$$

Thus

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \left[\sum_{k=1}^n \int_{E_k} f\right] = \sum_{n=1}^\infty \int_{E_n} f. \qquad \square$$

We leave it to the reader to use the countable additivity of integration to prove the following result regarding the continuity of integration: use as a pattern the proof of continuity of measure based on countable additivity of measure.

Theorem 21 (the Continuity of Integration) *Let f be integrable over E.*

(i) If $\{E_n\}_{n=1}^\infty$ is an ascending countable collection of measurable subsets of E, then

$$\int_{\bigcup_{n=1}^\infty E_n} f = \lim_{n\to\infty} \int_{E_n} f \tag{24}$$

(ii) If $\{E_n\}_{n=1}^\infty$ is a descending countable collection of measurable subsets of E, then

$$\int_{\bigcap_{n=1}^\infty E_n} f = \lim_{n\to\infty} \int_{E_n} f. \tag{25}$$

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37. Let f be a integrable function on E. Show that for each $\epsilon > 0$, there is a natural number N for which if $n \ge N$, then $\left|\int_{E_n} f\right| < \epsilon$ where $E_n = \{x \in E \mid |x| \ge n\}$.

38. For each of the two functions f on $[1, \infty)$ defined below, show that $\lim_{n\to\infty} \int_1^n f$ exists while f is not integrable over $[1, \infty)$. Does this contradict the continuity of integration?
    (i) Define $f(x) = (-1)^n/n$, for $n \le x < n+1$.
    (ii) Define $f(x) = (\sin x)/x$ for $1 \le x < \infty$.

39. Prove the theorem regarding the continuity of integration.

4.6 UNIFORM INTEGRABILITY: THE VITALI CONVERGENCE THEOREM

We conclude this first chapter on Lebesgue integration by establishing, for functions that are integrable over a set of finite measure, a criterion for justifying passage of the limit under the integral sign which is suggested by the following lemma and proposition.

Lemma 22 *Let E be a set of finite measure and $\delta > 0$. Then E is the disjoint union of a finite collection of sets, each of which has measure less than δ.*

Proof By the continuity of measure,

$$\lim_{n \to \infty} m(E \sim [-n, n]) = m(\emptyset) = 0.$$

Choose a natural number n_0 for which $m(E \sim [-n_0, n_0]) < \delta$. By choosing a fine enough partition of $[-n_0, n_0]$, express $E \cap [-n_0, n_0]$ as the disjoint union of a finite collection of sets, each of which has measure less than δ. □

Proposition 23 *Let f be a measurable function on E. If f is integrable over E, then for each $\epsilon > 0$, there is a $\delta > 0$ for which*

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon. \tag{26}$$

Conversely, in the case $m(E) < \infty$, if for each $\epsilon > 0$, there is a $\delta > 0$ for which (26) holds, then f is integrable over E.

Proof The theorem follows by establishing it separately for the positive and negative parts of f. We therefore suppose $f \geq 0$ on E. First assume f is integrable over E. Let $\epsilon > 0$. By the definition of the integral of a nonnegative integrable function, there is a measurable bounded function f_ϵ of finite support for which

$$0 \leq f_\epsilon \leq f \text{ on } E \text{ and } 0 \leq \int_E f - \int_E f_\epsilon < \epsilon/2.$$

Since $f - f_\epsilon \geq 0$ on E, if $A \subseteq E$ is measurable, then, by the linearity and additivity over domains of the integral,

$$\int_A f - \int_A f_\epsilon = \int_A [f - f_\epsilon] \leq \int_E [f - f_\epsilon] = \int_E f - \int_E f_\epsilon < \epsilon/2.$$

But f_ϵ is bounded. Choose $M > 0$ for which $0 \leq f_\epsilon < M$ on E_0. Therefore, if $A \subseteq E$ is measurable, then

$$\int_A f < \int_A f_\epsilon + \epsilon/2 \leq M \cdot m(A) + \epsilon/2.$$

Define $\delta = \epsilon/2M$. Then (26) holds for this choice of δ. Conversely, suppose $m(E) < \infty$ and for each $\epsilon > 0$, there is a $\delta > 0$ for which (26) holds. Let $\delta_0 > 0$ respond to the $\epsilon = 1$ challenge. Since $m(E) < \infty$, according to the preceding lemma, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$, each of which has measure less than δ. Therefore

$$\sum_{k=1}^{N} \int_{E_k} f < N.$$

By the additivity over domains of integration it follows that if h is a nonnegative measurable function of finite support and $0 \le h \le f$ on E, then $\int_E h < N$. Therefore f is integrable. □

Definition *A family $\mathcal{F}$ of measurable functions on E is said to be* **uniformly integrable over**[9] *E provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,*

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon. \tag{27}$$

Example Let g be a nonnegative integrable function over E. Define

$$\mathcal{F} = \left\{ f \mid f \text{ is measurable on } E \text{ and } |f| \le g \text{ on } E \right\}.$$

Then $\mathcal{F}$ is uniformly integrable. This follows from Proposition 23, with f replaced by g, and the observation that for any measurable subset A of E, by the monotonicity of integration, if f belongs to $\mathcal{F}$, then

$$\int_A |f| \le \int_A g.$$

Proposition 24 *Let $\{f_k\}_{k=1}^n$ be a finite collection of functions, each of which is integrable over E. Then $\{f_k\}_{k=1}^n$ is uniformly integrable.*

Proof Let $\epsilon > 0$. For $1 \le k \le n$, by Proposition 23, there is a $\delta_k > 0$ for which

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta_k, \text{ then } \int_A |f_k| < \epsilon. \tag{28}$$

Define $\delta = \min\{\delta_1, \ldots, \delta_n\}$. This δ responds to the ϵ challenge regarding the criterion for the collection $\{f_k\}_{k=1}^n$ to be uniformly integrable. □

Proposition 25 *Assume E has finite measure. Let the sequence of functions $\{f_n\}$ be uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E.*

Proof Let $\delta_0 > 0$ respond to the $\epsilon = 1$ challenge in the uniform integrability criteria for the sequence $\{f_n\}$. Since $m(E) < \infty$, by Lemma 22, we may express E as the disjoint union of a finite collection of measurable subsets $\{E_k\}_{k=1}^N$ such that $m(E_k) < \delta_0$ for $1 \le k \le N$. For any n, by the monotonicity and additivity over domains property of the integral,

$$\int_E |f_n| = \sum_{k=1}^{N} \int_{E_k} |f_n| < N.$$

[9] What is here called "uniformly integrable" is sometimes called "equiintegrable."

We infer from Fatou's Lemma that

$$\int_E |f| \leq \liminf \int_E |f_n| \leq N.$$

Thus $|f|$ is integrable over E. □

The Vitali Convergence Theorem *Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E.*

If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f.$

Proof Propositions 25 tells us that f is integrable over E and hence, by Proposition 15, is finite a.e. on E. Therefore, using Proposition 15 once more, by possibly excising from E a set of measure zero, we suppose the convergence is pointwise on all of E and f is real-valued. We infer from the integral comparison test and the linearity, monotonicity, and additivity over domains property of integration that, for any measurable subset A of E and any natural number n,

$$\begin{aligned} \left|\int_E f_n - \int_E f\right| &= \left|\int_E (f_n - f)\right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E\sim A} |f_n - f| + \int_A |f_n - f| \\ &\leq \int_{E\sim A} |f_n - f| + \int_A |f_n| + \int_A |f|. \end{aligned} \tag{29}$$

Let $\epsilon > 0$. By the uniform integrability of $\{f_n\}$, there is a $\delta > 0$ such that $\int_A |f_n| < \epsilon/3$ for any measurable subset of E for which $m(A) < \delta$. Therefore, by Fatou's Lemma, we also have $\int_A |f| \leq \epsilon/3$ for any measurable subset of A for which $m(A) < \delta$. Since f is real-valued and E has finite measure, Egoroff's Theorem tells us that there is a measurable subset E_0 of E for which $m(E_0) < \delta$ and $\{f_n\} \to f$ uniformly on $E \sim E_0$. Choose a natural number N such that $|f_n - f| < \epsilon/[3 \cdot m(E)]$ on $E \sim E_0$ for all $n \geq N$. Take $A = E_0$ in the integral inequality (29). If $n \geq N$, then

$$\begin{aligned} \left|\int_E f_n - \int_E f\right| &\leq \int_{E\sim E_0} |f_n - f| + \int_{E_0} |f_n| + \int_{E_0} |f_n| \\ &< \epsilon/[3 \cdot m(E)] \cdot m(E \sim E_0) + \epsilon/3 + \epsilon/3 \leq \epsilon. \end{aligned}$$

This completes the proof. □

The following theorem shows that the concept of uniform integrability is an essential ingredient in the justification, for a sequence $\{h_n\}$ of nonnegative functions on a set of finite measure that converges pointwise to $h \equiv 0$, of passage of the limit under the integral sign.

Theorem 26 *Let E be of finite measure. Suppose $\{h_n\}$ is a sequence of nonnegative integrable functions that converges pointwise a.e. on E to $h \equiv 0$. Then*

$$\lim_{n\to\infty} \int_E h_n = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable over } E.$$

Proof If $\{h_n\}$ is uniformly integrable, then, by the Vitali Convergence Theorem, $\lim_{n\to\infty} \int_E h_n = 0$. Conversely, suppose $\lim_{n\to\infty} \int_E h_n = 0$. Let $\epsilon > 0$. We may choose a natural number N for which $\int_E h_n < \epsilon$ if $n \geq N$. Therefore, since each $h_n \geq 0$ on E,

$$\text{if } A \subseteq E \text{ is measurable and } n \geq N, \text{ then } \int_A h_n < \epsilon. \tag{30}$$

According to Propositions 23 and 24, the finite collection $\{h_n\}_{n=1}^{N-1}$ is uniformly integrable over E. Let δ respond to the ϵ challenge regarding the criterion for the uniform integrability of $\{h_n\}_{n=1}^{N-1}$. We infer from (30) that δ also responds to the ϵ challenge regarding the criterion for the uniform integrability of $\{h_n\}_{n=1}^{\infty}$. □

PROBLEMS

40. Let f be integrable over $\mathbf{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbf{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

41. Show that Proposition 25 is false if $E = \mathbf{R}$.

42. Show that Theorem 26 is false without the assumption that the h_n's are nonnegative.

43. Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E. Show that for any α and β, the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable over E.

44. Let f be integrable over $\mathbf{R}$ and $\epsilon > 0$. Establish the following three approximation properties.
    (i) There is a simple function η on $\mathbf{R}$ which has finite support and $\int_{\mathbf{R}} |f - \eta| < \epsilon$ (Hint: First verify this if f is nonnegative.]
    (ii) There is a step function s on $\mathbf{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbf{R}} |f - s| < \epsilon$. (Hint: Apply part (i) and Problem 18 of Chapter 3.)
    (iii) There is a continuous function g on $\mathbf{R}$ which vanishes outside a bounded set and $\int_{\mathbf{R}} |f - g| < \epsilon$.

45. Let f be integrable over E. Define $\hat{f}$ to be the extension of f to all of $\mathbf{R}$ obtained by setting $\hat{f} \equiv 0$ outside of E. Show that $\hat{f}$ is integrable over $\mathbf{R}$ and $\int_E f = \int_{\mathbf{R}} \hat{f}$. Use this and part (i) and (iii) of the preceding problem to show that for $\epsilon > 0$, there is a simple function η on E and a continuous function g on E for which $\int_E |f - \eta| < \epsilon$ and $\int_E |f - g| < \epsilon$.

46. (Riemann-Lebesgue) Let f be integrable over $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos nx\, dx = 0.$$

(Hint: First show this for f is a step function that vanishes outside a closed, bounded interval and then use the approximation property (ii) of Problem 44.)

47. Let f be integrable over $(-\infty, \infty)$.

(i) Show that for each t,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f(x+t)\, dx.$$

(ii) Let g be a bounded measurable function on $\mathbf{R}$. Show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x)\cdot[f(x) - f(x+t)] = 0.$$

(Hint: First show this, using uniform continuity of f on $\mathbf{R}$, if f is continuous and vanishes outside a bounded set. Then use the approximation property (iii) of Problem 44.)

48. Let f be integrable over E and g be a bounded measurable function on E. Show that $f \cdot g$ is integrable over E.

49. Let f be integrable over $\mathbf{R}$. Show that the following four assertions are equivalent:

(i) $f = 0$ a.e on $\mathbf{R}$.

(ii) $\int_{\mathbf{R}} fg = 0$ for every bounded measurable function g on $\mathbf{R}$.

(iii) $\int_A f = 0$ for every measurable set A.

(iv) $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

50. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left|\int_A f\right| < \epsilon.$$

51. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E\cap\mathcal{U}} |f| < \epsilon.$$

52. (a) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_0^1 |f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?

(b) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is continuous on $[0, 1]$ and has $|f| \leq 1$ on $[0, 1]$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?

(c) Let $\mathcal{F}$ be the family of functions f on $[0, 1]$, each of which is integrable over $[0, 1]$ and has $\int_a^b |f| \leq b - a$ for all $[a, b] \subseteq [0, 1]$. Is $\mathcal{F}$ uniformly integrable over $[0, 1]$?