CHAPTER 2

Lebesgue Measure

Contents

2.1 INTRODUCTION

The Riemann integral of a bounded function over a closed, bounded interval is defined using approximations of the function that are associated with partitions of its domain into finite collections of subintervals. The generalization of the Riemann integral to the Lebesgue integral will be achieved by using approximations of the function that are associated with decompositions of its domain into finite collections of sets which we call Lebesgue measurable. Each interval is Lebesgue measurable. The richness of the collection of Lebesgue measurable sets provides better upper and lower approximations of a function, and therefore of its integral, than are possible by just employing intervals. This leads to a larger class of functions that are Lebesgue integrable over very general domains and an integral that has better properties. For instance, under quite general circumstances we will prove that if a sequence of functions converges pointwise to a limiting function, then the integral of the limit function is the limit of the integrals of the approximating functions. In this chapter we establish the basis for the forthcoming study of Lebesgue measurable functions and the Lebesgue integral: the basis is the concept of measurable set and the Lebesgue measure of such a set.

The length $\ell(I)$ of an interval I is defined to be the difference of the endpoints of I if I is bounded, and ∞ if I is unbounded. Length is an example of a *set function*, that is, a function that associates an extended real number to each set in a collection of sets. In the case of length, the domain is the collection of all intervals. In this chapter we extend the set function length to a large collection of sets of real numbers. For instance, the "length" of an open set will be the sum of the lengths of the countable number of open intervals of which it is composed. However, the collection of sets consisting of intervals and open sets is still too limited for our purposes. We construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure** which is denoted by m.

The collection of Lebesgue measurable sets is a σ-algebra[1] which contains all open sets and all closed sets. The set function m possesses the following three properties.

The measure of an interval is its length Each nonempty interval I is Lebesgue measurable and

$$m(I) = \ell(I).$$

Measure is translation invariant If E is Lebesgue measurable and y is any number, then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and

$$m(E + y) = m(E).$$

Measure is countably additivity over countable disjoint unions of sets[2] If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (see page 48). In fact, there is not even a set function defined for all sets of real numbers that possesses the first two properties and is finitely additive (see Theorem 18). We respond to this limitation by constructing a set function on a very rich class of sets that does possess the above three properties. The construction has two stages.

We first construct a set function called **outer–measure**, which we denote by m^*. It is defined for any set, and thus, in particular, for any interval. The outer measure of an interval is its length. Outer measure is translation invariant. However, outer measure is not finitely additive. But it is countably subadditive in the sense that if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, disjoint or not, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

The second stage in the construction is to determine what it means for a set to be **Lebesgue measurable** and show that the collection of Lebesgue measurable sets is a σ-algebra containing the open and closed sets. We then restrict the set function m^* to the collection of Lebesgue measurable sets, denote it by m, and prove m is countably additive. We call m **Lebesgue measure**.

[1]A collection of subsets of $\mathbf{R}$ is called a σ-algebra provided it contains $\mathbf{R}$ and is closed with respect to the formation of complements and countable unions; by De Morgan's Identities, such a collection is also closed with respect to the formation of countable intersections.

[2]For a collection of sets to be disjoint we mean what is sometimes called pairwise disjoint, that is, that each pair of sets in the collection has empty intersection.

PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.
2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.
3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.
4. A set function c, defined on all subsets of $\mathbf{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.

2.2 LEBESGUE OUTER MEASURE

Let I be a nonempty interval of real numbers. We define its length, $\ell(I)$, to be ∞ if I is unbounded and otherwise define its length to be the difference of its endpoints. For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A, that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure**[3] of A, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is **monotone** in the sense that

$$\text{if } A \subseteq B, \text{ then } m^*(A) \leq m^*(B).$$

Example A countable set has outer measure zero. Indeed, let C be a countable set enumerated as $C = \{c_k\}_{k=1}^{\infty}$. Let $\epsilon > 0$. For each natural number k, define $I_k = (c_k - \epsilon/2^{k+1},\ c_k + \epsilon/2^{k+1})$. The countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covers C. Therefore

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

This inequality holds for each $\epsilon > 0$. Hence $m^*(E) = 0$.

Proposition 1 *The outer measure of an interval is its length.*

[3]There is a general concept of outer measure, which will be considered in Part III. The set function m^* is a particular example of this general concept, which is properly identified as Lebesgue outer measure on the real line. In Part I, we refer to m^* simply as outer measure.

Proof We begin with the case of a closed, bounded interval $[a, b]$. Let $\epsilon > 0$. Since the open interval $(a-\epsilon, b+\epsilon)$ contains $[a, b]$ we have $m^*([a, b]) \leq \ell((a-\epsilon, b+\epsilon)) = b-a+2\epsilon$. This holds for any $\epsilon > 0$. Therefore $m^*([a, b]) \leq b-a$. It remains to show that $m^*([a, b]) \geq b-a$. But this is equivalent to showing that if $\{I_k\}_{k=1}^{\infty}$ is any countable collection of open, bounded intervals covering $[a, b]$, then

$$\sum_{k=1}^{\infty} \ell(I_k) \geq b-a. \tag{1}$$

By the Heine-Borel Theorem,[4] any collection of open intervals covering $[a, b]$ has a finite subcollection that also covers $[a, b]$. Choose a natural number n for which $\{I_k\}_{k=1}^{n}$ covers $[a, b]$. We will show that

$$\sum_{k=1}^{n} \ell(I_k) \geq b-a, \tag{2}$$

and therefore (1) holds. Since a belongs to $\bigcup_{k=1}^{n} I_k$, there must be one of the I_k's that contains a. Select such an interval and denote it by (a_1, b_1). We have $a_1 < a < b_1$. If $b_1 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq b_1 - a_1 > b-a.$$

Otherwise, $b_1 \in [a, b)$, and since $b_1 \notin (a_1, b_1)$, there is an interval in the collection $\{I_k\}_{k=1}^{n}$, which we label (a_2, b_2), distinct from (a_1, b_1), for which $b_1 \in (a_2, b_2)$; that is, $a_2 < b_1 < b_2$. If $b_2 \geq b$, the inequality (2) is established since

$$\sum_{k=1}^{n} \ell(I_k) \geq (b_1 - a_1) + (b_2 - a_2) = b_2 - (a_2 - b_1) - a_1 > b_2 - a_1 > b-a.$$

We continue this selection process until it terminates, as it must since there are only n intervals in the collection $\{I_k\}_{k=1}^{n}$. Thus we obtain a subcollection $\{(a_k, b_k)\}_{k=1}^{N}$ of $\{I_k\}_{k=1}^{n}$ for which

$$a_1 < a,$$

while

$$a_{k+1} < b_k \text{ for } 1 \leq k \leq N-1,$$

and, since the selection process terminated,

$$b_N > b.$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} \ell(I_k) &\geq \sum_{k=1}^{N} \ell((a_i, b_i)) \\ &= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \cdots + (b_1 - a_1) \\ &= b_N - (a_N - b_{N-1}) - \ldots - (a_2 - b_1) - a_1 \\ &> b_N - a_1 > b-a.\end{aligned}$$

[4]See page 18.