Therefore f is integrable over E and, by (2) and (3),

$$\left|\int_E [f_n - f]\right| < \epsilon \text{ for all } n \geq N.$$

The proof is complete. □

We leave the proof of the following corollary as an exercise.

Corollary 2 *Let $\{h_n\}$ be a sequence of nonnegative integrable functions on E. Suppose $\{h_n(x)\} \to 0$ for almost all x in E. Then*

$$\lim_{n\to\infty} \int_E h_n = 0 \text{ if and only if } \{h_n\} \text{ is uniformly integrable and tight over } E.$$

PROBLEMS

1. Prove Corollary 2.
2. Let $\{f_k\}_{k=1}^n$ be a finite family of functions, each of which is integrable over E. Show that $\{f_k\}_{k=1}^n$ is uniformly integrable and tight over E.
3. Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable and tight over E. Show that for any α and β, $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable and tight over E.
4. Let $\{f_n\}$ be a sequence of measurable functions on E. Show that $\{f_n\}$ is uniformly integrable and tight over E if and only if for each $\epsilon > 0$, there is a measurable subset E_0 of E that has finite measure and a $\delta > 0$ such that for each measurable subset A of E and index n,

$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \epsilon.$$

5. Let $\{f_n\}$ be a sequence of integrable functions on $\mathbf{R}$. Show that $\{f_n\}$ is uniformly integrable and tight over $\mathbf{R}$ if and only if for each $\epsilon > 0$, there are positive numbers r and δ such that for each open subset $\mathcal{O}$ of $\mathbf{R}$ and index n,

$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{ then } \int_{\mathcal{O}} |f_n| < \epsilon.$$

5.2 CONVERGENCE IN MEASURE

We have considered sequences of functions that converge uniformly, that converge pointwise, and that converge pointwise almost everywhere. To this list we add one more mode of convergence that has useful relationships both to pointwise convergence almost everywhere and to forthcoming criteria for justifying the passage of the limit under the integral sign.

Definition *Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E for which f and each f_n is finite a.e. on E. The sequence $\{f_n\}$ is said to* **converge in measure** *on E to f provided for each $\eta > 0$,*

$$\lim_{n\to\infty} m\left\{x \in E \mid |f_n(x) - f(x)| > \eta\right\} = 0.$$

When we write $\{f_n\} \to f$ in measure on E we are implicitly assuming that f and each f_n is measurable, and finite a.e. on E. Observe that if $\{f_n\} \to f$ uniformly on E, and f is a real-valued measurable function on E, then $\{f_n\} \to f$ in measure on E since for $\eta > 0$, the set $\{x \in E \mid |f_n(x) - f(x)| > \eta\}$ is empty for n sufficiently large. However, we also have the following much stronger result.

Proposition 3 *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f and f is finite a.e. on E. Then $\{f_n\} \to f$ in measure on E.*

Proof First observe that f is measurable since it is the pointwise limit almost everywhere of a sequence of measurable functions. Let $\eta > 0$. To prove convergence in measure we let $\epsilon > 0$ and seek an index N such that

$$m\left\{x \in E \;\middle|\; |f_n(x) - f(x)| > \eta\right\} < \epsilon \text{ for all } n \geq N. \tag{4}$$

Egoroff's Theorem tells us that there is a measurable subset F of E with $m(E \sim F) < \epsilon$ such that $\{f_n\} \to f$ uniformly on F. Thus there is an index N such that

$$|f_n - f| < \eta \text{ on } F \text{ for all } n \geq N.$$

Thus, for $n \geq N$, $\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq E \sim F$ and so (4) holds for this choice of N. □

The above proposition is false if E has infinite measure. The following example shows that the converse of this proposition also is false.

Example Consider the sequence of subintervals of [0, 1], $\{I_n\}_{n=1}^{\infty}$, which has initial terms listed as

$$[0, 1], [0, 1/2], [1/2, 1], [0, 1/3], [1/3, 2/3], [2/3, 1],$$
$$[0, 1/4], [1/4, 1/2], [1/2, 3/4], [3/4, 1] \ldots$$

For each index n, define f_n to be the restriction to [0, 1] of the characteristic function of I_n. Let f be the function that is identically zero on [0, 1]. We claim that $\{f_n\} \to f$ in measure. Indeed, observe that $\lim_{n \to \infty} \ell(I_n) = 0$ since for each natural number m,

$$\text{if } n > 1 + \cdots + m = \frac{m(m+1)}{2}, \text{ then } \ell(I_n) < 1/m.$$

Thus, for $0 < \eta < 1$, since $\{x \in E \mid |f_n(x) - f(x)| > \eta\} \subseteq I_n$,

$$0 \leq \lim_{n \to \infty} m\left\{x \in E \;\middle|\; |f_n(x) - f(x)| > \eta\right\} \leq \lim_{n \to \infty} \ell(I_n) = 0.$$

However, it is clear that there is no point x in [0, 1] at which $\{f_n(x)\}$ converges to $f(x)$ since for each point x in [0, 1], $f_n(x) = 1$ for infinitely many indices n, while $f(x) = 0$.

Theorem 4 (Riesz) *If $\{f_n\} \to f$ in measure on E, then there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.*

Proof By the definition of convergence in measure, there is a strictly increasing sequence of natural numbers $\{n_k\}$ for which

$$m\left\{x\in E \mid |f_j(x)-f(x)|>1/k\right\}<1/2^k \text{ for all } j\geq n_k.$$

For each index k, define

$$E_k=\left\{x\in E \mid |f_{n_k}-f(x)|>1/k\right\}.$$

Then $m(E_k)<1/2^k$ and therefore $\sum_{k=1}^{\infty} m(E_k)<\infty$. The Borel-Cantelli Lemma tells us that for almost all $x\in E$, there is an index $K(x)$ such that $x\notin E_k$ if $k\geq K(x)$, that is,

$$|f_{n_k}(x)-f(x)|\leq 1/k \text{ for all } k\geq K(x).$$

Therefore

$$\lim_{k\to\infty} f_{n_k}(x)=f(x).$$

□

Corollary 5 *Let $\{f_n\}$ be a sequence of nonnegative integrable functions on E. Then*

$$\lim_{n\to\infty}\int_E f_n=0 \tag{5}$$

if and only if

$$\{f_n\}\to 0 \textit{ in measure on } E \textit{ and } \{f_n\} \textit{ is uniformly integrable and tight over } E. \tag{6}$$

Proof First assume (5). Corollary 2 tells us that $\{f_n\}$ is uniformly integrable and tight over E. To show that $\{f_n\}\to 0$ in measure on E, let $\eta>0$. By Chebychev's Inequality, for each index n,

$$m\left\{x\in E \mid f_n>\eta\right\}\leq\frac{1}{\eta}\cdot\int_E f_n.$$

Thus,

$$0\leq\lim_{n\to\infty} m\left\{x\in E \mid f_n>\eta\right\}\leq\frac{1}{\eta}\cdot\lim_{n\to\infty}\int_E f_n=0.$$

Hence $\{f_n\}\to 0$ in measure on E.

To prove the converse, we argue by contradiction. Assume (6) holds but (5) fails to hold. Then there is some $\epsilon_0>0$ and a subsequence $\{f_{n_k}\}$ for which

$$\int_E f_{n_k}\geq\epsilon_0 \text{ for all } k.$$

However, by Theorem 4, a subsequence of $\{f_{n_k}\}$ converges to $f\equiv 0$ pointwise almost everywhere on E and this subsequence is uniformly integrable and tight so that, by the Vitali Convergence Theorem, we arrive at a contradiction to the existence of the above ϵ_0. This completes the proof. □

PROBLEMS

6. Let $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Show that $\{f_n\} \to g$ in measure on E if and only if $f = g$ a.e. on E.
7. Let E have finite measure, $\{f_n\} \to f$ in measure on E and g be a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure, and use this to show that $\{f_n^2\} \to f^2$ in measure. Infer from this that if $\{g_n\} \to g$ in measure, then $\{f_n \cdot g_n\} \to f \cdot g$ in measure.
8. Show that Fatou's Lemma, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem remain valid if "pointwise convergence *a.e.*" is replaced by "convergence in measure."
9. Show that Proposition 3 does not necessarily hold for sets E of infinite measure.
10. Show that linear combinations of sequences that converge in measure on a set of finite measure also converge in measure.
11. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable on E for which f and each f_n is finite a.e. on E. Prove that $\{f_n\} \to f$ in measure on E if and only if every subsequence of $\{f_n\}$ has in turn a further subsequence that converges to f pointwise a.e. on E.
12. Show that a sequence $\{a_j\}$ of real numbers converges to a real number if $|a_{j+1} - a_j| \leq 1/2^j$ for all j by showing that the sequence $\{a_j\}$ must be Cauchy.
13. A sequence $\{f_n\}$ of measurable functions on E is said to be **Cauchy in measure** provided given $\eta > 0$ and $\epsilon > 0$ there is an index N such that for all $m, n \geq N$,
$$m\left\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\right\} < \epsilon.$$
Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f on E to which the sequence $\{f_n\}$ converges in measure. (Hint: Choose a strictly increasing sequence of natural numbers $\{n_j\}$ such that for each index j, if $E_j = \{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > 1/2^j\}$, then $m(E_j) < 1/2^j$. Now use the Borel-Cantelli Lemma and the preceding problem.)
14. Assume $m(E) < \infty$. For two measurable functions g and h on E, define
$$\rho(g, h) = \int_E \frac{|g - h|}{1 + |g - h|}.$$
Show that $\{f_n\} \to f$ in measure on E if and only if $\lim_{n \to \infty} \rho(f_n, f) = 0$.

5.3 CHARACTERIZATIONS OF RIEMANN AND LEBESGUE INTEGRABILITY

Lemma 6 *Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of functions, each of which is integrable over E, such that $\{\varphi_n\}$ is increasing while $\{\psi_n\}$ is decreasing on E. Let the function f on E have the property that*
$$\varphi_n \leq f \leq \psi_n \text{ on } E \text{ for all } n.$$
If
$$\lim_{n \to \infty} \int_E [\psi_n - \varphi_n] = 0,$$
then

$\{\varphi_n\} \to f$ pointwise a.e. on E, $\{\psi_n\} \to f$ pointwise a.e. on E, f is integrable over E,