## Mass-Energy Relation

The most significant relation obtained from the special theory of relativity concerns mass and energy. As in classical mechanics, the kinetic energy of a moving body is defined as equal to the work done on the body in bringing it to that state of motion from its state of rest. Therefore, the kinetic energy dT acquired by the body when it is displaced by a force $\mathbf{f}$ through a distance dr is given by

$$dT = \mathbf{f} \cdot dr, \qquad (3.37)$$

so that the rate of increase of kinetic energy is

$$\frac{dT}{dt} = \mathbf{f} \cdot \frac{dr}{dt} = \mathbf{f} \cdot \bar{V},$$

where $\bar{V}$ is the velocity of the particle.

or $$\frac{dT}{dt} = \left[\frac{d}{dt}(m\,V)\right] \cdot V = \frac{dm}{dt} V \cdot V + m\frac{dV}{dt} \cdot V$$

$$= V^2 \frac{dm}{dt} + m\,V \frac{dV}{dt}. \tag{3.38}$$

But $$m = \frac{m_0}{\sqrt{1 - V^2/c^2}}.$$

Differentiating m with respect to t, we get

$$\frac{dm}{dt} = \frac{m_0}{c^2(1 - V^2/c^2)^{3/2}} V \frac{dV}{dt} = \frac{m\,V}{c^2(1 - V^2/c^2)} \frac{dV}{dt}$$

or $$\frac{dV}{dt} = \frac{c^2\,(1 - V^2/c^2)}{m\,V} \frac{dm}{dt}.$$

Substituting this expression for dV/dt in equation (3.38), we get

$$\frac{dT}{dt} = V^2 \frac{dm}{dt} + c^2\,(1 - V^2/c^2) \frac{dm}{dt} = c^2 \frac{dm}{dt}. \tag{3.39}$$

Integrating with respect to t, we get

$$T = m\,c^2 + K,$$

where K is a constant

or $$T = \frac{m_0}{\sqrt{1 - V^2/c^2}} c^2 + K.$$

If the kinetic energy of the particle at rest is taken as zero, then K must be equal to $- m_0\,c^2$. Hence

$$T = m\,c^2 - m_0\,c^2 = \frac{m_0}{\sqrt{1 - V^2/c^2}} c^2 - m_0\,c^2. \tag{3.40}$$

This gives us an expression for the kinetic energy of the particle. For $V \ll$ ... expanding $(1 - V^2/c^2)^{-1/2}$ by binomial theorem, we obtain

$$T = m_0\,c^2\,(1 + \frac{1}{2} V^2/c^2 + \frac{3}{8} V^4/c^4 + \ldots$$

interact with the material system under observation. This cannot be avoided. Moreover, the very act of observation and communication will also cause interaction between the observer and the frame. This last objection can be overcome by assuming that such interactions are negligible, i.e., the system and the frame of reference are very heavy as compared with the objects carrying the information.

## Examples of Equivalence of Mass and Energy

We shall now consider some important examples of the equivalence of mass and energy which have been confirmed experimentally.

### *Binding Energy*

We know that a nucleus is composed of protons and neutrons. It may therefore be expected that the mass of a nucleus would be equal to the sum of the masses of its constituent particles. A survey of nuclear mass tables, however, shows that this is not true. The mass of a stable nucleus is always less than the sum of the masses of its constituent particles. This difference in mass is easily accounted for by assuming that the mass difference $\Delta m$ is changed into energy $\Delta E$ ($\Delta E = c^2 \Delta m$) which is used to bind the nucleons together. This is called the binding energy of the nucleus and is evidently equal to the energy required to break up the nucleus into its constituent particles. We shall illustrate this by an example. Helium nucleus, ${}^4_2He$, consists of two protons and two neutrons and its mass is 4.003874 amu, where 1 amu is 1/12 of the mass of ${}^{12}_6C$ and is equal to $1.66033 \times 10^{-27}$ kg. The mass of a proton is 1.007646 amu while that of a neutron is 1.009034 amu. Therefore

the mass of (2 protons + 2 neutrons)

$= (2 \times 1.007646 + 2 \times 1.009034)$ amu

$= 4.03336$ amu.

This is greater than the mass of the helium nucleus; the mass difference being

0.02949 amu. The nucleus is therefore stable, as an energy of 27.46 MeV, equivalent to the above mass difference, is required to break it into its constituent particles. This is the binding energy of the helium nucleus.

**Problem**

Show that the binding energy of the deuteron nucleus is 2.226 MeV.

## *Natural Radioactivity*

Some radioactive nuclides disintegrate spontaneously, say, by the emission of $\alpha$-particles. It is found that for such substances, the sum of the masses of the products of disintegration is less than the mass of the disintegrating nuclide. The corresponding difference in mass is released in the form of energy. We shall illustrate this by considering the disintegration of radium into radon and helium:

$$^{226}_{88}Ra \rightarrow ^{222}_{86}Rn + ^{4}_{2}He.$$

The mass of $^{226}_{88}Ra$ nucleus (226.09600 amu) is greater than the sum of the masses of nuclei of $^{222}_{86}Rn$ (222.08690 amu) and $^{4}_{2}He$ (4.003874 amu) by 0.00523 amu. Hence radium is unstable against $\alpha$-decay. The energy released in disintegration is c times the mass difference so that $E = \Delta m\, c^2 = 4.87$ MeV.

## *Nuclear Reactions*

Energy may also be released in nuclear reactions which are particularly suited for verifying the law $E = mc^2$. The nuclear masses are known with great precision and, more important, the velocities of the final particles are large enough so that the diminution in mass can be appreciable. For instance, consider the reaction

$$^{7}_{3}Li + ^{1}_{1}H \rightarrow 2\,^{4}_{2}He$$

in which two particles ($^{4}_{2}He$ nuclei) per reaction are produced when lithium ($^{7}_{3}Li$) is bombarded with monoenergetic protons. We have

Mass of $^{7}_{3}Li$ = 7.018232 amu