4-Force and Equation of Motion

4 components of force

In classical mechanics, Newton's second law of motion may be taken as giving the definition of force. That is, if the momentum p of a particle changes, then the rate of change of momentum of the particle, dp/dt, is called the force **F** on the particle:

$$\mathbf{F} = \frac{dp}{dt}$$

or $$F_i = \frac{dp_i}{dt}, \quad i = 1, 2, 3.$$

In special relativity, we are free to choose any definition of force which reduces to the classical expression for classical situations. From a mathematical point of view, it is convenient to introduce the concept of force by defining a 4-vector F_μ, called 4-force, as the rate of change of 4-momentum with respect to proper time:

$$F_\mu = \frac{dp_\mu}{d\tau}. \tag{3.16}$$

To check whether the space components of F_μ reduce to classical expressions for classical situations, we first write down the space components of F_μ explicitly. We have

$$F_1 = \frac{dp_1}{d\tau} = \frac{dp_1}{dt}\frac{dt}{d\tau}$$

$$= \frac{1}{\sqrt{1 - V^2/c^2}} \frac{d}{dt}(m\,V_x) = \frac{f_x}{\sqrt{}} = \gamma(V) f_x$$

$$= m_0\,\gamma(V) \frac{d}{dt}[\gamma(V)\,V_x]. \tag{3.17a}$$

Similarly, we can obtain

$$F_2 = m_0\,\gamma(V) \frac{d}{dt}[\gamma(V)\,V_y] = \frac{f_y}{\sqrt{}} \tag{3.17b}$$

$$F_3 = m_0\,\gamma(V) \frac{d}{dt}[\gamma(V)\,V_z]. = \frac{f_z}{\sqrt{}} \tag{3.17c}$$

It may be stressed that the velocity V of a particle is, in general, a variable quantity. We note that for $V \ll c$, the spatial components F_1, F_2, F_3 of F_μ reduce to the classical expressions for the components of force, thus vindicating our definition of 4-force. Moreover, as both sides of equation (3.16) transform as 4-vectors, this equation is covariant under a Lorentz transformation.

Hence equations (3.17 a,b,c) can be taken as expressing the correct law of motion in relativistic mechanics. However, 3-force is no more an absolute quantity.

The fourth component of F_μ is given by

$$F_4 = \frac{dp_4}{d\tau} = \frac{dp_4}{dt}\frac{dt}{d\tau} = \gamma(V)\frac{d}{dt}(imc)$$

$$= ic\,\gamma(V)\frac{dm}{dt}. \tag{3.17d}$$

Show That

When we are considering a system of particles in special relativity, a difficulty arises in using equation (3.16) because we have to use a different proper time for each particle. A different definition of force, which is more convenient when we are considering a system of particles, is given through the relation

$$f_\mu = \frac{dp_\mu}{dt}. \tag{3.18}$$

It is this definition of force which is usually used in the literature. From the above relation, we have

$$f_1 = \frac{dp_1}{dt} = \frac{d}{dt}(m\,V_x). \tag{3.19a}$$

Since m is the mass of the particle when it is moving with a velocity V, the expression mV_x may be considered as the x-component of the 3-momentum provided we interpret m as mass of the particle when it is moving with a velocity V. Then the x-component of the 3-force may be written as

$$f_x = \frac{d}{dt}(m\,V_x).$$

Thus we obtain

$$f_1 = f_x.$$

Similarly, we can show that f_2 and f_3 are equal to f_y and f_z respectively, so that we may write

$$\mathbf{f} = \frac{d}{dt}(m\,\mathbf{V}). \tag{3.19b}$$

Moreover, $$f_4 = \frac{dp_4}{dt} = \frac{d}{dt}(imc) = ic\frac{dm}{dt}. \tag{3.19c}$$

Thus:

$$f_\mu = (f_1, f_2, f_3, f_4) = (f_x, f_y, f_z, ic\frac{dm}{dt}) = (\mathbf{f}, ic\frac{dm}{dt}). \tag{3.19d}$$

Problem

Prove that $$\mathbf{f}\cdot\mathbf{V} = c^2\frac{dm}{dt}. \tag{3.20}$$

Solution

We have shown that

$$V_\mu V_\mu = -c^2. \tag{3.3'}$$

Differentiating with respect to t, we get

$$V_\mu \frac{d}{dt}(V_\mu) = 0$$

or $$V_\mu \frac{d}{dt}(m_0 V_\mu) = 0 \qquad p_\mu = m_0 V_\mu$$

or $$V_\mu f_\mu = 0. \tag{3.21}$$

or $$V_1 f_1 + V_2 f_2 + V_3 f_3 + V_4 f_4 = 0$$

or $$\sum_{a=1,2,3} \gamma(V)\, V_a f_a - c^2 \gamma(V) \frac{dm}{dt} = 0,$$

where we have used equations (3.2) and the relations $f_{1,2,3} = f_{x,y,z}$.

or $$\mathbf{f}\cdot\mathbf{V} = c^2 \frac{dm}{dt}. \tag{3.20'}$$

This gives the required relation. Equation (3.19d) may then be written as

$$f_\mu = \left(\mathbf{f}, \frac{i}{c}\, \mathbf{f}\cdot\mathbf{V}\right). \tag{3.22}$$

$P = \vec{f}\cdot\vec{v} = c^2 \frac{dm}{dt} = -ic\left(-ic\frac{dm}{dt}\right) = -ic f_4$

Problem

Show that power is $-ic$ times the fourth component of f_μ.

transformation law for

It may be noted that

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{dp_\mu}{dt}\frac{dt}{d\tau} = \gamma(V)\, f_\mu \tag{3.23}$$

This gives us the relation between two different definitions of force. It may be stressed that while F_μ is a 4-vector, f_μ is not. Therefore, although equation (3.18) can be taken as a definition of force, it cannot be the equation of motion in relativistic mechanics because it transformation.

The transformation law for $f_\mu = (f_x, f_y, f_z, f_4)$ may be obtained as follows:

Since F_μ is a 4-vector, in going from one inertial frame to another, its components will transform as

$$F_1' = \gamma\,(F_1 + i\,\frac{v}{c}\,F_4) \tag{3.24a}$$

$$F_2' = F_2 \tag{3.24b}$$

$$F_3' = F_3 \tag{3.24c}$$

$$F_4' = \gamma\,(F_4 - i\,\frac{v}{c}\,F_1). \tag{3.24d}$$

By using equations (3.23), (3.17a,d), (3.6) and a relation similar to equation (3.23) in the primed frame, equation (3.24a) can be written as

$$\begin{aligned} f_x' &= A\,(f_x - v\,\frac{dm}{dt}) \\ &= A\,(f_x - \frac{v}{c^2}\,\mathbf{f}\cdot\mathbf{V}), \end{aligned} \tag{3.25}$$

where we have used the result obtained in equation (3.21). From equations (3.6) and (3.25), we get

$$\begin{aligned} f_x' &= (f_x - v\,\frac{dm}{dt})/(1 - \frac{v}{c^2}\,V_x) \\ &= (f_x - \frac{v}{c^2}\,\mathbf{f}\cdot\mathbf{V})/(1 - \frac{v}{c^2}\,V_x). \end{aligned} \tag{3.26a}$$

Similarly, we obtain

$$f_y' = f_y/[\gamma\,(1 - \frac{v}{c^2}\,V_x)] \tag{3.26b}$$

$$f_z' = f_z/[\gamma\,(1 - \frac{v}{c^2}\,V_x)]. \tag{3.26c}$$

In particular, if the velocity of the particle in the S-frame is parallel to the x-axis: $\mathbf{V} = (V, 0, 0)$, and the force is also acting in the same direction: $\mathbf{f} =$

$i\,m'c\,\gamma(V)' = \gamma\,[\,i\,m c\,\gamma(V) - i\,\frac{v}{c}\,\gamma(V)\,f_x)$

$m'c'^2 = m c^2 - v f_x$

$\vec{f}'\cdot\vec{V}' = A\,(\vec{f}\cdot\vec{V} - v f_x)$