Position 4-Vector and 4-Velocity

We start our search for correct equations of motion by considering the motion of a particle moving with a velocity $V = (V_x, V_y, V_z)$. The position (x_1, x_2, x_3) of this particle at any instant t can be described by a point $P(x_1, x_2, x_3, x_4 = ict)$ in a frame of reference K in 4-dimensional space-time continuum. The vector drawn from the origin O(0, 0, 0, 0) of K to the position of the particle is called its position 4-vector. In this case, the coordinates x_1, x_2, x_3, x_4 of the particle are also the components of this 4-vector. The position 4-vector is therefore usually denoted by x_μ. The rate of change of position 4-vector x_μ of the particle with respect to its proper time will also be a 4-vector because the proper time is an invariant quantity. This is called 4-velocity and is denoted by V_μ:

$$V_\mu = \frac{dx_\mu}{d\tau}. \tag{3.1}$$

Problem

Can we obtain a velocity 4-vector by differentiating x_μ with respect to t?

4-velocity Components

Equation (3.1) may be written as

$$V_\mu = \frac{dx_\mu}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - V^2/c^2}}\frac{dx_\mu}{dt} = \gamma(V)\frac{dx_\mu}{dt}.$$

Let us write explicitly the components of the 4-velocity V_μ. We have

$$V_1 = \gamma(V)\frac{dx_1}{dt} = \gamma(V)\,V_x. \tag{3.2a}$$

Similarly,

$$V_2 = \gamma(V)\,V_y \tag{3.2b}$$

$$V_3 = \gamma(V)\,V_z \tag{3.2c}$$

$$V_4 = \gamma(V)\frac{dx_4}{dt} = ic\,\gamma(V). \tag{3.2d}$$

We may therefore write

$$V_\mu = (V_1, V_2, V_3, V_4) = \gamma(V)\,(\mathbf{V}, ic) = \frac{1}{\sqrt{1 - V^2/c^2}}\,(\mathbf{V}, ic). \qquad (3.2e)$$

Results

1- For $V/c \to 0$, the function $\gamma(V) \to 1$, and the first three components of the 4-velocity reduce to the components of the Newtonian 3-velocity. The magnitude of the 4-velocity is given by

2

2- $$V_\mu V_\mu = V_1^2 + V_2^2 + V_3^2 + V_4^2 = \gamma(V)\,(V_x^2 + V_y^2 + V_z^2) - c^2\,\gamma^2(V)$$

$$= \gamma^2(V)\,(V^2 - c^2) = \frac{1}{1 - V^2/c^2}(V^2 - c^2) = -c^2. \qquad (3.3)$$

This is a constant quantity as the magnitude of every 4-vector must be. This also shows that the 4-velocity is a time-like vector.

Since V_μ is a 4-vector, its components V_1, V_2, V_3, V_4 transform in the same way as x_1, x_2, x_3, x_4, i.e., as x, y, z, ict. Hence the transformation equations for velocity are given by

using Transformation eqs for velocity derive 3 velocity components

$$V_1' = \gamma(v)\,(V_1 + i\tfrac{v}{c} V_4) \qquad (3.4a)$$

$$V_2' = V_2 \qquad (3.4b)$$

$$V_3' = V_3 \qquad (3.4c)$$

$$V_4' = \gamma(v)\,(V_4 - i\tfrac{v}{c} V_1). \qquad (3.4d)$$

Here V and V′ are the speeds of the particle in the two frames of reference, v is the relative speed of the two frames and $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$. To find the transformation laws for V_x, V_y, V_z, we substitute the expressions for V_1, V_2, V_3 from equations (3.2) in equations (3.4) and use similar relations between V_1', V_2', V_3' and V_x', V_y', V_z'. We then obtain

$$V_x' = A\,(V_x - v) \qquad (3.5a)$$

$$V_y' = \gamma(V)\,V_y/\gamma(V') = A\,V_y/\gamma(v) \qquad (3.5b)$$

$$V_z' = \gamma(V)\,V_z/\gamma(V') = A\,V_z/\gamma(v), \qquad (3.5c)$$

where $A = \gamma(v)\,\gamma(V)/\gamma(V')$. $\qquad (3.5d)$

Now substituting the expressions for V_1, V_4 and V'_4 from equations (3.
(3.2d) and from a primed equation similar to equation (3.2d) in equa
(3.4d), we get

$$1 = \gamma(v)\,\gamma(V)\,(1 - \frac{v}{c^2} V_x)/\gamma(V').$$

Using equation (3.5d), this can be written as

$$1 = A\,(1 - \frac{v}{c^2} V_x). \tag{3.6}$$

Substituting the expression for A from equation (3.6) in equations (3.5a–
c), we get

$$V'_x = \frac{V_x - v}{1 \cancel{+}\!- \frac{v}{c^2} V_x} \tag{3.7a}$$

$$V'_y = \frac{V_y}{\gamma\,(1 \cancel{+}\!- \frac{v}{c^2} V_x)} \tag{3.7b}$$

$$V'_z = \frac{V_z}{\gamma\,(1 \cancel{+}\!- \frac{v}{c^2} V_x)}, \tag{3.7c}$$

where, as already pointed out, γ stands for $\gamma(v)$.

The last three equations give the transformation law for components of 3-velocity.

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