3.2 MINKOWSKI'S FOUR-DIMENSIONAL SPACE–TIME WORLD

The negative results of the Michelson–Morley experiment point out that velocity does not behave exactly like the mathematical object called vector. The resultant of two parallel vectors of magnitude c and v is, by the vector operation of addition, equal to a vector of magnitude $v + c$. The resultant vector, which results from the addition of the velocity of light to the velocity of moving reference system, still gives the same velocity of light, and so it does not behave like a mathematical vector.

Minkowski[1] imparted great elegance to the theory by introducing a fictitious construct, four-dimensional space called *Minkowski world*. The basic elements of the physical world are *events*, which are described by the time when occurred and by the space where occurred. Thus an event has four coordinates for its location, i.e., t, x, y, z. The main crux of Minkowski's theory is that these four numbers must be treated as an ordered set of four components of a vector and a suitable algebra of such four-dimensional vectors must be built to describe the relationships between them.

Consider two reference systems S and S', the x-axis parallel to the x'-axis and the origins O and O' coinciding at $t = 0$. The system S' moves with velocity v with respect to S. Supposing that at time $t = 0$, a spherical electromagnetic wave is sent into free space from the common origin. At any subsequent time, the equations of wave front given in the two systems S and S' are

$$c^2t^2 = x^2 + y^2 + z^2 \tag{3.15}$$

$$c^2t'^2 = x'^2 + y'^2 + z'^2 \tag{3.16}$$

These are the equations of a sphere moving out with speed c in all directions. (ct, x, y, z) are the time and coordinates measured by the observer in S whereas (ct', x', y', z'), the time and coordinates measured by the observer in S'. These two conflicting descriptions of the same physical phenomenon can be reconciled if we adopt the following linear transformations

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{3.17(a)}$$

$$x' = \gamma(x - vt) \tag{3.17(b)}$$

$$y' = y;\ z' = z \tag{3.17(c)}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$.

The mathematical implications of these transformations suggest that time cannot be considered a scalar under it and the three space coordinates do not behave like the components of a classical three-dimensional vector. But when $v << c$, Lorentz contraction factor $\gamma \to 1$ and the usual classical meanings are ascribed to t, x, y, and z.

We see that the expression $c^2t^2 - x^2 - y^2 - z^2$ is invariant under Lorentz transformation. This is called the square length of the interval, whose four components are (ct, x, y, z). Representing the coordinates of a point P by x_0, x_1, x_2, x_3 in a four-dimensional continuum, the location of a point P, called *world point*, with respect to a fixed origin O is determined by

$$R = \sum_{\mu=0}^{3} x_\mu I_\mu \qquad I_\mu I_\nu = \delta_{\mu\nu} \tag{3.18}$$

The linear transformation

$$x'_\mu = \sum_{\mu=0}^{3} \alpha_{\mu\nu} x_\nu \tag{3.19(a)}$$

will be orthogonal if the coefficients are subjected to the following condition

$$\sum_{\mu=0}^{3} \alpha_{\mu\rho}\alpha_{\mu\nu} = \delta_{\rho\nu} \tag{3.19(b)}$$

3.2.1 Notation

We shall denote the components of three-dimensional vectors by Latin indices i, j, k whereas the Greek indices μ, ν, ρ, λ will be used to denote the components of a four-vector. In accordance with the usual convention of Tensor analysis, the repetition of an index implies summation over the range of the index. Thus, in a three-dimensional space

$$\mathbf{A}\cdot\mathbf{B} = A_iB_i = A_1B_1 + A_2B_2 + A_3B_3 \quad [3.20(a)]$$

whereas when A and B stand for four-vectors, we have

$$AB = A_\mu B_\mu = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 \quad [3.20(b)]$$

3.3 LORENTZ TRANSFORMATION AS ROTATION IN FOUR-SPACE

We will again show that classical transformation equations (Galilean) can be replaced by Lorentz transformations that are based on the invariance of the velocity of light and not on the assumption of universal time and absolute space (invariant length of scale). In deriving these equations, we assume the principle of relativity as fundamental, implying that all inertial frames are equivalent. Furthermore, it is assumed that space is homogeneous, i.e., all points of space are equivalent. The resulting equations must be linear equations.

The postulate of the constancy of velocity of light c will be obeyed by the group of transformations that leave the equation of wave front invariant

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \quad (3.21)$$

The space in which the line element, Eq. (3.21), is written is pseudo-Euclidean. Euclidean manifold is a manifold with all coefficients equal to 1, i.e., +1, +1, +1, +1 in four dimensions. These are called the signature of space. This may be achieved by choosing

$$x_0 = ict$$

$$x_1 = x$$

$$x_2 = y$$

$$x_3 = z$$

Therefore,

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0'^2 + x_1'^2 + x_2'^2 + x_3'^2 \quad (3.22)$$

But the above equation is invariant to the group of rotations in four-space and hence we construe that transformation that connects the coordinates of an event in S system to the coordinates of the same event in S', corresponding to a rotation in a four-dimensional space x_0, x_1, x_2, x_3. The four-space consists of three dimensions of ordinary space with a fourth imaginary dimension proportional to time. This space is called *world space or Minkowski space*.