43. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by $I(\theta) = N^2 \sin^2 k/k^2$, where $k = (\pi Nd \sin\theta)/\lambda$ and d is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-6}}^{10^{-6}} I(\theta)\, d\theta$ emerging from the grating.

44. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)\, dx$. Compare your result to the actual value. Can you explain the discrepancy?

45. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

46. Sketch the graph of a continuous function on $[0, 2]$ for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

47. If f is a positive function and $f''(x) < 0$ for $a \leqslant x \leqslant b$, show that

$$T_n < \int_a^b f(x)\, dx < M_n$$

48. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)\, dx$.

49. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

50. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x)\, dx$ we dealt with a function f defined on a finite interval $[a, b]$ and we assumed that f does not have an infinite discontinuity (see Section 5.2). In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral. One of the most important applications of this idea, probability distributions, will be studied in Section 8.5.

Type 1: Infinite Intervals

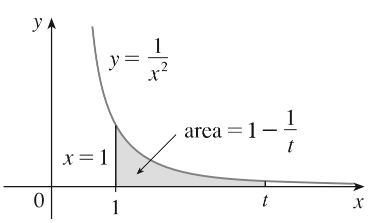

FIGURE 1

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$

Notice that $A(t) < 1$ no matter how large t is chosen.

We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = 1$$

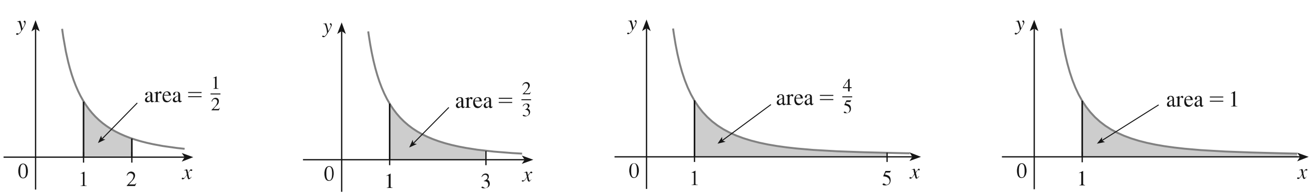

FIGURE 2

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

Any of the improper integrals in Definition 1 can be interpreted as an area provided that f is a positive function. For instance, in case (a) if $f(x) \geqslant 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region $S = \{(x, y) \mid x \geqslant a,\ 0 \leqslant y \leqslant f(x)\}$ in Figure 3 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of f from a to t.

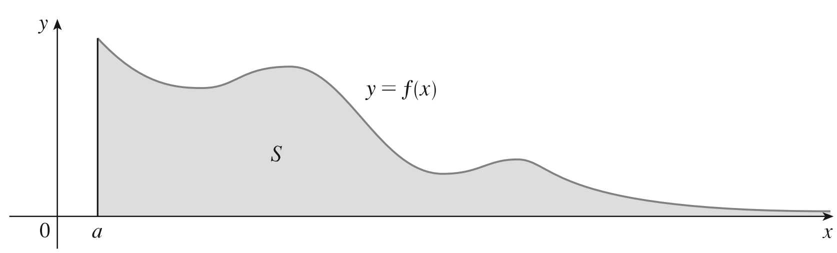

FIGURE 3

EXAMPLE 1 Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

SOLUTION According to part (a) of Definition 1, we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \ln 1) = \lim_{t\to\infty} \ln t = \infty$$

The limit does not exist as a finite number and so the improper integral $\int_1^\infty (1/x)\,dx$ is divergent. ■

Let's compare the result of Example 1 with the example given at the beginning of this section:

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

Geometrically, this says that although the curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$, the region under $y = 1/x^2$ to the right of $x = 1$ (the shaded region in Figure 4) has finite area whereas the corresponding region under $y = 1/x$ (in Figure 5) has infinite area. Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$ but $1/x^2$ approaches 0 faster than $1/x$. The values of $1/x$ don't decrease fast enough for its integral to have a finite value.

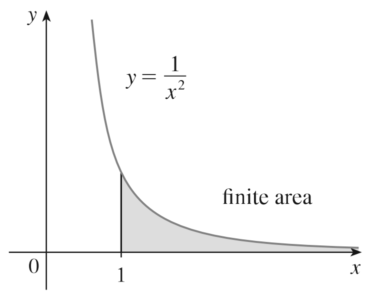

FIGURE 4
$\int_1^\infty (1/x^2)\,dx$ converges

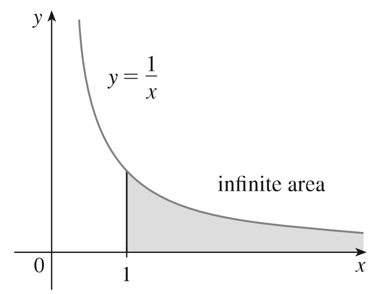

FIGURE 5
$\int_1^\infty (1/x)\,dx$ diverges

EXAMPLE 2 Evaluate $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

SOLUTION Using part (b) of Definition 1, we have

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx$$

We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = xe^x\Big]_t^0 - \int_t^0 e^x\,dx$$
$$= -te^t - 1 + e^t$$

TEC In Module 7.8 you can investigate visually and numerically whether several improper integrals are convergent or divergent.

We know that $e^t \to 0$ as $t \to -\infty$, and by l'Hospital's Rule we have

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$
$$= \lim_{t\to-\infty} (-e^t) = 0$$

Therefore

$$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} (-te^t - 1 + e^t)$$
$$= -0 - 1 + 0 = -1$$ ■

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

SOLUTION It's convenient to choose $a = 0$ in Definition 1(c):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

We must now evaluate the integrals on the right side separately:

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1}x\Big]_0^t$$

$$= \lim_{t\to\infty} (\tan^{-1}t - \tan^{-1}0) = \lim_{t\to\infty} \tan^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1}x\Big]_t^0$$

$$= \lim_{t\to-\infty} (\tan^{-1}0 - \tan^{-1}t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since both of these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

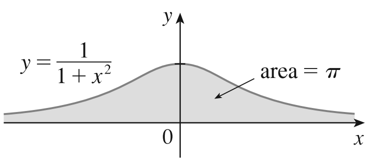

FIGURE 6

Since $1/(1 + x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x-axis (see Figure 6). ■

EXAMPLE 4 For what values of p is the integral

$$\int_1^{\infty} \frac{1}{x^p}\,dx$$

convergent?

SOLUTION We know from Example 1 that if $p = 1$, then the integral is divergent, so let's assume that $p \neq 1$. Then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$. Therefore

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} \qquad \text{if } p > 1$$

and so the integral converges. But if $p < 1$, then $p - 1 < 0$ and so

$$\frac{1}{t^{p-1}} = t^{1-p} \to \infty \qquad \text{as } t \to \infty$$

and the integral diverges. ■

We summarize the result of Example 4 for future reference:

2 $$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

Type 2: Discontinuous Integrands

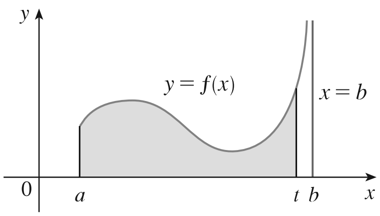

FIGURE 7

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b. Let S be the unbounded region under the graph of f and above the x-axis between a and b. (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.) The area of the part of S between a and t (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x)\,dx$$

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

Parts (b) and (c) of Definition 3 are illustrated in Figures 8 and 9 for the case where $f(x) \geqslant 0$ and f has vertical asymptotes at a and c, respectively.

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

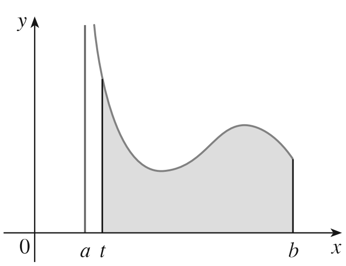

FIGURE 8

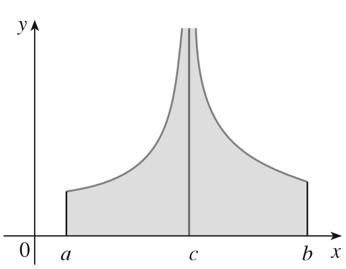

FIGURE 9

EXAMPLE 5 Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

SOLUTION We note first that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$. Since the infinite discontinuity occurs at the left

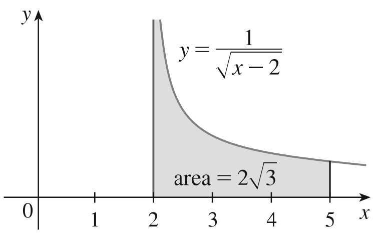

FIGURE 10

endpoint of $[2, 5]$, we use part (b) of Definition 3:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5$$

$$= \lim_{t\to 2^+} 2\big(\sqrt{3} - \sqrt{t-2}\,\big) = 2\sqrt{3}$$

Thus the given improper integral is convergent and, since the integrand is positive, we can interpret the value of the integral as the area of the shaded region in Figure 10. ■

EXAMPLE 6 Determine whether $\int_0^{\pi/2} \sec x\, dx$ converges or diverges.

SOLUTION Note that the given integral is improper because $\lim_{x\to(\pi/2)^-} \sec x = \infty$. Using part (a) of Definition 3 and Formula 14 from the Table of Integrals, we have

$$\int_0^{\pi/2} \sec x\, dx = \lim_{t\to(\pi/2)^-} \int_0^t \sec x\, dx = \lim_{t\to(\pi/2)^-} \ln\left|\sec x + \tan x\right|\Big]_0^t$$

$$= \lim_{t\to(\pi/2)^-} [\ln(\sec t + \tan t) - \ln 1] = \infty$$

because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$. Thus the given improper integral is divergent. ■

EXAMPLE 7 Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible.

SOLUTION Observe that the line $x = 1$ is a vertical asymptote of the integrand. Since it occurs in the middle of the interval $[0, 3]$, we must use part (c) of Definition 3 with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where
$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln\left|x-1\right|\Big]_0^t$$

$$= \lim_{t\to 1^-} \big(\ln\left|t-1\right| - \ln\left|-1\right|\big) = \lim_{t\to 1^-} \ln(1-t) = -\infty$$

because $1 - t \to 0^+$ as $t \to 1^-$. Thus $\int_0^1 dx/(x-1)$ is divergent. This implies that $\int_0^3 dx/(x-1)$ is divergent. [We do not need to evaluate $\int_1^3 dx/(x-1)$.] ■

WARNING If we had not noticed the asymptote $x = 1$ in Example 7 and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln\left|x-1\right|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

From now on, whenever you meet the symbol $\int_a^b f(x)\, dx$ you must decide, by looking at the function f on $[a, b]$, whether it is an ordinary definite integral or an improper integral.

EXAMPLE 8 $\displaystyle\int_0^1 \ln x\, dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since

$\lim_{x \to 0^+} \ln x = -\infty$. Thus the given integral is improper and we have

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x \, dx = x \ln x \Big]_t^1 - \int_t^1 dx$$

$$= 1 \ln 1 - t \ln t - (1 - t) = -t \ln t - 1 + t$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t} = \lim_{t \to 0^+} \frac{1/t}{-1/t^2} = \lim_{t \to 0^+} (-t) = 0$$

Therefore $$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} (-t \ln t - 1 + t) = -0 - 1 + 0 = -1$$

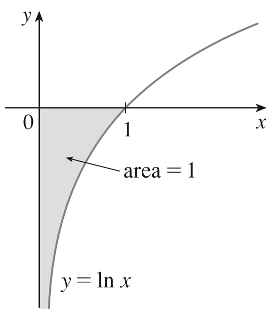

FIGURE 11

Figure 11 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1. ■

■ A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

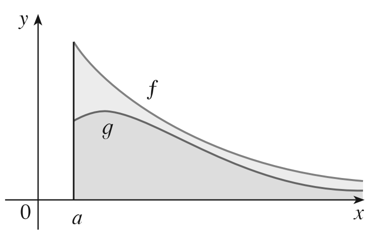

FIGURE 12

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible. If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$. And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. [Note that the reverse is not necessarily true: If $\int_a^\infty g(x)\,dx$ is convergent, $\int_a^\infty f(x)\,dx$ may or may not be convergent, and if $\int_a^\infty f(x)\,dx$ is divergent, $\int_a^\infty g(x)\,dx$ may or may not be divergent.]

EXAMPLE 9 Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

SOLUTION We can't evaluate the integral directly because the antiderivative of e^{-x^2} is not an elementary function (as explained in Section 7.5). We write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

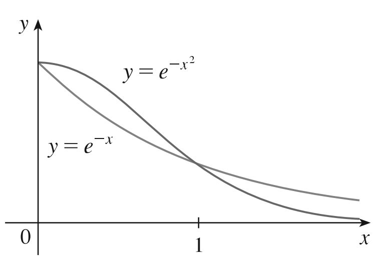

FIGURE 13

and observe that the first integral on the right-hand side is just an ordinary definite integral. In the second integral we use the fact that for $x \geqslant 1$ we have $x^2 \geqslant x$, so $-x^2 \leqslant -x$ and therefore $e^{-x^2} \leqslant e^{-x}$. (See Figure 13.) The integral of e^{-x} is easy to evaluate:

$$\int_1^\infty e^{-x} dx = \lim_{t \to \infty} \int_1^t e^{-x} dx = \lim_{t \to \infty} (e^{-1} - e^{-t}) = e^{-1}$$

Therefore, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent. ■

Table 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

In Example 9 we showed that $\int_0^\infty e^{-x^2}\,dx$ is convergent without computing its value. In Exercise 72 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral, as we will see in Section 8.5; using the methods of multivariable calculus it can be shown that the exact value is $\sqrt{\pi}/2$. Table 1 illustrates the definition of an improper integral by showing how the (computer-generated) values of $\int_0^t e^{-x^2}\,dx$ approach $\sqrt{\pi}/2$ as t becomes large. In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

Table 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

Table 2 illustrates the divergence of the integral in Example 10. It appears that the values are not approaching any fixed number.

7.8 EXERCISES

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^2 \frac{x}{x - 1}\,dx$ (b) $\displaystyle\int_0^\infty \frac{1}{1 + x^3}\,dx$

(c) $\displaystyle\int_{-\infty}^\infty x^2 e^{-x^2}\,dx$ (d) $\displaystyle\int_0^{\pi/4} \cot x\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_0^{\pi/4} \tan x\,dx$ (b) $\displaystyle\int_0^\pi \tan x\,dx$

(c) $\displaystyle\int_{-1}^1 \frac{dx}{x^2 - x - 2}$ (d) $\displaystyle\int_0^\infty e^{-x^3}\,dx$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.

(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.

(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–40 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\displaystyle\int_3^\infty \frac{1}{(x - 2)^{3/2}}\,dx$

6. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1 + x}}\,dx$

7. $\displaystyle\int_{-\infty}^0 \frac{1}{3 - 4x}\,dx$

8. $\displaystyle\int_1^\infty \frac{1}{(2x + 1)^3}\,dx$

9. $\displaystyle\int_2^\infty e^{-5p}\,dp$

10. $\displaystyle\int_{-\infty}^0 2^r\,dr$

11. $\displaystyle\int_0^\infty \frac{x^2}{\sqrt{1 + x^3}}\,dx$

12. $\displaystyle\int_{-\infty}^\infty (y^3 - 3y^2)\,dy$

13. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

14. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$

15. $\displaystyle\int_0^\infty \sin^2\alpha\,d\alpha$

16. $\displaystyle\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta$

17. $\displaystyle\int_1^\infty \frac{1}{x^2 + x}\,dx$

18. $\displaystyle\int_2^\infty \frac{dv}{v^2 + 2v - 3}$

19. $\displaystyle\int_{-\infty}^0 ze^{2z}\,dz$

20. $\displaystyle\int_2^\infty ye^{-3y}\,dy$

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

22. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

23. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4 + 4}\,dz$

24. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

25. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy$

26. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}}$

27. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

28. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5 - x}}\,dx$

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x + 2}}$

30. $\displaystyle\int_{-1}^2 \frac{x}{(x + 1)^2}\,dx$

31. $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx$

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

33. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

34. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

35. $\displaystyle\int_0^{\pi/2} \tan^2\theta\,d\theta$

36. $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$

37. $\displaystyle\int_0^1 r\ln r\,dr$

38. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

39. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx$

40. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx$

41–46 Sketch the region and find its area (if the area is finite).

41. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant e^{-x}\}$

42. $S = \{(x, y) \mid x \leqslant 0,\ 0 \leqslant y \leqslant e^{x}\}$

43. $S = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/(x^3 + x)\}$

44. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant xe^{-x}\}$

45. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

46. $S = \{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\}$

47. (a) If $g(x) = (\sin^2 x)/x^2$, use your calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2, 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

48. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use your calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?
(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$

51. $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

55. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

56. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

by the same method as in Exercise 55.

57–59 Find the values of p for which the integral converges and evaluate the integral for those values of p.

57. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

58. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

59. $\displaystyle\int_0^1 x^p \ln x\,dx$

60. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.
(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.
(c) Prove your guess using mathematical induction.

61. (a) Show that $\int_{-\infty}^\infty x\,dx$ is divergent.
(b) Show that

$$\lim_{t\to\infty} \int_{-t}^t x\,dx = 0$$

This shows that we can't define

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty} \int_{-t}^t f(x)\,dx$$

62. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

63. We know from Example 1 that the region $\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid with finite volume.

64. Use the information and data in Exercise 6.4.33 to find the work required to propel a 1000-kg space vehicle out of the earth's gravitational field.

65. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 6.4.33) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.