exist in nature. The question then arises, how would it be possible to find out whether a given frame of reference is inertial or not? For an inertial frame, an object, at rest or moving with uniform velocity in it, should not change its state in the absence of an external force. But, in practice, the gravitational forces are always present to influence the behaviour of the object under consideration and it is not possible to check what would happen if these forces were not present. However, a frame of reference in which the first law holds at least approximately, can be determined by considering the behaviour of an object far removed from all other bodies. For instance, if we use a frame of reference which is rigidly attached to the fixed stars, then the position of the heavenly bodies relative to this frame is very nearly uniform, indicating that this frame is approximately an inertial frame of reference. The departure from uniformity can reasonably be accounted for as due to the influence of the stars upon one another.

We shall now show that a frame of reference which is in uniform motion relative to an inertial frame is also inertial. For this purpose let us first find the relation connecting two frames of reference moving uniformly with respect to each other.

The Galilean Transformation

Consider a frame of reference S′ which is coincident with an inertial frame S at time t = 0 and is moving with a constant velocity v relative to S in

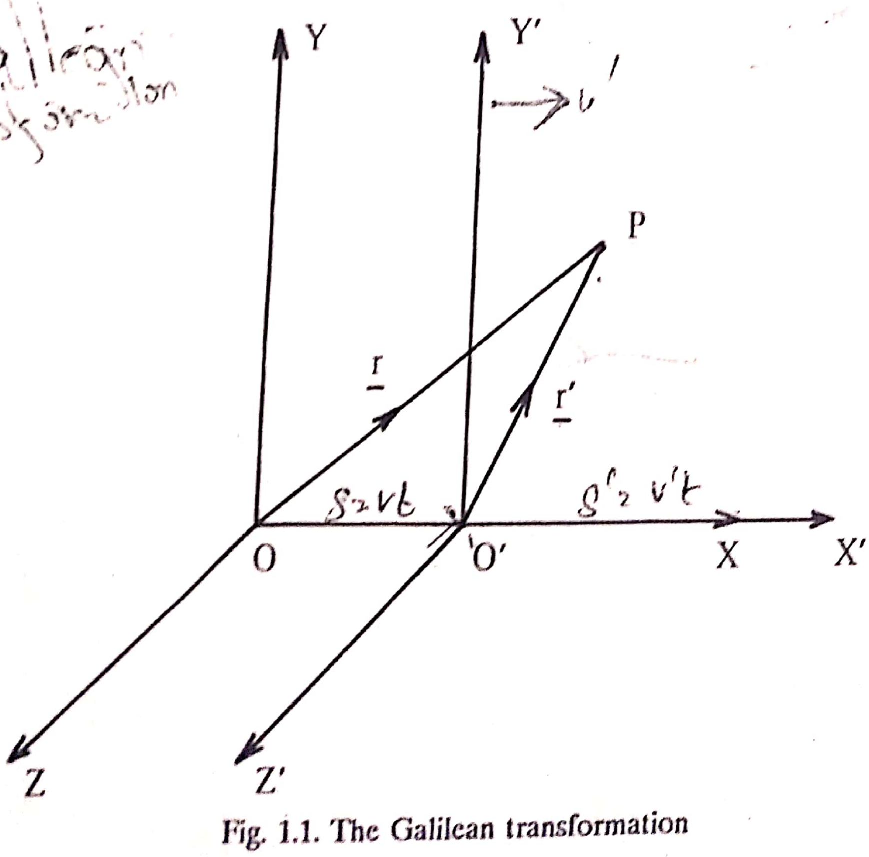

Fig. 1.1. The Galilean transformation

the positive direction of their common x-axes as shown in Fig. 1.1. We shall call S and S′ **standard frames** or **frames in standard configuration.** Let O′ be the position of the origin of S′ after a time t when an event occurs at a point P. Let **r** and **r′** be the position vectors of the point P with respect to O and O′ respectively. Then from Fig. 1.1:

$$\overrightarrow{O'P} = \overrightarrow{OP} - \overrightarrow{OO'}$$

or $\quad \mathbf{r}' = \mathbf{r} - \mathbf{v}\,t. \qquad (1.1)$

Let x, y, z, and x′, y′, z′ be the coordinates of the same point P in the frames S and S′. Then

$$(x', y', z') = (x, y, z) - (v, 0, 0)\,t$$

OP = O′P + OO′

so that

$$x' = x - v\,t, \tag{1.2a}$$

$$y' = y, \tag{1.2b}$$

$$z' = z. \tag{1.2c}$$

In classical physics, time is absolute so that it is unaffected by a motion of the frame of reference. Therefore if t′ denotes the time at which the event at P is observed from the frame S′, then

$$t' = t. \tag{1.2d}$$

That is, the observers in the frames S and S′ will assign the same time to any given event.

Equations (1.2) show how the space and time coordinates of an event in two standard frames of reference S and S′ are related to each other. This particular transformation of coordinates is called the Galilean transformation for frames in standard configuration. It is found to be valid experimentally whenever the relative velocity between S and S′ is small as compared to the velocity of light.

We are now well equipped to show that all frames of reference which are in uniform motion relative to an inertial frame are themselves inertial. To prove this, consider two standard frames of reference S and S′, such that S is an inertial frame. We have to show that S′ is also inertial.

According to the Galilean transformation, the position vectors of any particle in the two frames are related at any time t by equation (1.1). Differentiating this equation with respect to t, we get

$$\frac{d\mathbf{r}'}{dt} = \frac{d\mathbf{r}}{dt} - \mathbf{v}.$$

Since $t' = t$, we may write the above equation as

$$\frac{d\mathbf{r}'}{dt'} = \frac{d\mathbf{r}}{dt} - \mathbf{v}$$

or $$\mathbf{V}' = \mathbf{V} - \mathbf{v},$$

where $\mathbf{V}$ and $\mathbf{V}'$ denote respectively the velocities of the same particle as measured in the frames S and S′. Therefore if the velocity of a particle is constant in the frame S, then its velocity in S′ will also be constant. That is if S is an inertial frame, the frame S′ will also be inertial. This proves the above statement.

Covariance of Newton's Second Law of Motion

It is tempting to assume, and we do assume, that Newton's second law of motion, expressed symbolically by the vector equation

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{V}) = m\frac{d^2\mathbf{r}}{dt^2}, \tag{1.3}$$

where $\mathbf{V}$ is the velocity of a particle of mass m and $\mathbf{F}$ is the force acting on it, is valid in an inertial frame. Clearly, for $\mathbf{F} = 0$, equation (1.3) is simply a symbolic expression for the first law.

We next show that the vector equation of motion has the same form in all inertial frames of reference.

Consider two standard frames of reference S and S′ related by the Galilean transformation (1.1). On differentiation with respect to t′, equation (1.1) yields

$$\frac{d^2\mathbf{r}'}{dt'^2} = \frac{d^2\mathbf{r}}{dt^2},$$

where we have also used the fact that $t' = t$ and that the relative velocity $\mathbf{v}$ is constant. Since force and mass are taken as absolute quantities in Newtonian mechanics they should have the same values in all inertial frames of reference:

$$\mathbf{F}' = \mathbf{F} \quad \text{and} \quad m' = m.$$

Substituting the expressions for **F**, m and $\dfrac{d^2\mathbf{r}}{dt^2}$ in equation (1.3), we get

$$\mathbf{F}' = m' \frac{d^2\mathbf{r}'}{dt'^2}.$$

This shows that the vector equation expressing Newton's second law of motion does not change in form in going from one inertial frame to another. This fact is expressed by stating that Newton's second law of motion or the corresponding equation is **covariant** under a Galilean transformation.

In general, if a law (or the equation expressing that law) does not change in going from one inertial frame of reference to another, it is said to be **covariant** under the transformation connecting the inertial frames. On the other hand, if a quantity or an expression remains unaltered in going from one inertial frame to another, it is said to be **invariant**. Thus in Newtonian mechanics, the force **F**, the mass m and the time t are invariant quantities whereas the position **r** is not invariant as it changes to **r**$'$ in going from one inertial frame to another. However, the equation

$$\mathbf{F} = m \frac{d^2\mathbf{r}}{dt^2}, \qquad (1.3')$$

does not change in such a transformation and is therefore covariant under the Galilean transformation connecting the inertial frames.

Newton's Second Law and Absolute Motion in Space

All the mechanical phenomena are governed by Newton's second law and the equation expressing this law has the same form in all inertial frames. This covariance has a great physical significance as identical mechanical experiments performed in any two inertial frames should yield identical results in both the frames and therefore it should not be possible to distinguish between different inertial frames by means of mechanical experiments. We would illustrate it by an example.

First of all, we note that if the rotation of th