# 7 Techniques of Integration

USDA

The photo shows a screw-worm fly, the first pest effectively eliminated from a region by the sterile insect technique without pesticides. The idea is to introduce into the population sterile males that mate with females but produce no offspring. In Exercise 7.4.67 you will evaluate an integral that relates the female insect population to time.

**BECAUSE OF THE FUNDAMENTAL THEOREM** of Calculus, we can integrate a function if we know an antiderivative, that is, an indefinite integral. We summarize here the most important integrals that we have learned so far.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

In this chapter we develop techniques for using these basic integration formulas to obtain indefinite integrals of more complicated functions. We learned the most important method of integration, the Substitution Rule, in Section 5.5. The other general technique, integration by parts, is presented in Section 7.1. Then we learn methods that are special to particular classes of functions, such as trigonometric functions and rational functions.

Integration is not as straightforward as differentiation; there are no rules that absolutely guarantee obtaining an indefinite integral of a function. Therefore we discuss a strategy for integration in Section 7.5.

## 7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts*.

The Product Rule states that if $f$ and $g$ are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

**1**

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

**2**

$$\int u\,dv = uv - \int v\,du$$

**EXAMPLE 1** Find $\int x \sin x\,dx$.

**SOLUTION USING FORMULA 1** Suppose we choose $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. (For $g$ we can choose *any* antiderivative of $g'$.) Thus, using Formula 1, we have

$$\begin{aligned}\int x \sin x\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= x(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

It's wise to check the answer by differentiating it. If we do so, we get $x \sin x$, as expected.

It is helpful to use the pattern:

$$u = \square \qquad dv = \square$$
$$du = \square \qquad v = \square$$

**SOLUTION USING FORMULA 2** Let

$$u = x \qquad dv = \sin x\, dx$$

Then

$$du = dx \qquad v = -\cos x$$

and so

$$\int x \sin x\, dx = \int \overbrace{x}^{u}\ \overbrace{\sin x\, dx}^{dv} = \overbrace{x}^{u}\ \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\ \overbrace{dx}^{du}$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C$$

■

**NOTE** Our aim in using integration by parts is to obtain a simpler integral than the one we started with. Thus in Example 1 we started with $\int x \sin x\, dx$ and expressed it in terms of the simpler integral $\int \cos x\, dx$. If we had instead chosen $u = \sin x$ and $dv = x\, dx$, then $du = \cos x\, dx$ and $v = x^2/2$, so integration by parts gives

$$\int x \sin x\, dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\, dx$$

Although this is true, $\int x^2 \cos x\, dx$ is a more difficult integral than the one we started with. In general, when deciding on a choice for $u$ and $dv$, we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\, dx$ can be readily integrated to give $v$.

**EXAMPLE 2** Evaluate $\int \ln x\, dx$.

**SOLUTION** Here we don't have much choice for $u$ and $dv$. Let

$$u = \ln x \qquad dv = dx$$

Then

$$du = \frac{1}{x}\, dx \qquad v = x$$

Integrating by parts, we get

$$\int \ln x\, dx = x \ln x - \int x \frac{dx}{x}$$

It's customary to write $\int 1\, dx$ as $\int dx$.

$$= x \ln x - \int dx$$

Check the answer by differentiating it.

$$= x \ln x - x + C$$

Integration by parts is effective in this example because the derivative of the function $f(x) = \ln x$ is simpler than $f$. ■

**EXAMPLE 3** Find $\int t^2 e^t\,dt$.

**SOLUTION** Notice that $t^2$ becomes simpler when differentiated (whereas $e^t$ is unchanged when differentiated or integrated), so we choose

$$u = t^2 \qquad dv = e^t\,dt$$

Then $$du = 2t\,dt \qquad v = e^t$$

Integration by parts gives

$$\int t^2 e^t\,dt = t^2 e^t - 2\int te^t\,dt \tag{3}$$

The integral that we obtained, $\int te^t\,dt$, is simpler than the original integral but is still not obvious. Therefore we use integration by parts a second time, this time with $u = t$ and $dv = e^t\,dt$. Then $du = dt$, $v = e^t$, and

$$\begin{aligned}\int te^t\,dt &= te^t - \int e^t\,dt\\ &= te^t - e^t + C\end{aligned}$$

Putting this in Equation 3, we get

$$\begin{aligned}\int t^2 e^t\,dt &= t^2 e^t - 2\int te^t\,dt\\ &= t^2 e^t - 2(te^t - e^t + C)\\ &= t^2 e^t - 2te^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C\end{aligned}$$ ■

**EXAMPLE 4** Evaluate $\int e^x \sin x\,dx$.

An easier method, using complex numbers, is given in Exercise 50 in Appendix H.

**SOLUTION** Neither $e^x$ nor $\sin x$ becomes simpler when differentiated, but we try choosing $u = e^x$ and $dv = \sin x\,dx$ anyway. Then $du = e^x\,dx$ and $v = -\cos x$, so integration by parts gives

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx \tag{4}$$

The integral that we have obtained, $\int e^x \cos x\,dx$, is no simpler than the original one, but at least it's no more difficult. Having had success in the preceding example integrating by parts twice, we persevere and integrate by parts again. This time we use $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx \tag{5}$$

At first glance, it appears as if we have accomplished nothing because we have arrived at $\int e^x \sin x\,dx$, which is where we started. However, if we put the expression for $\int e^x \cos x\,dx$ from Equation 5 into Equation 4 we get

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

Figure 1 illustrates Example 4 by showing the graphs of $f(x) = e^x \sin x$ and $F(x) = \frac{1}{2}e^x(\sin x - \cos x)$. As a visual check on our work, notice that $f(x) = 0$ when $F$ has a maximum or minimum.

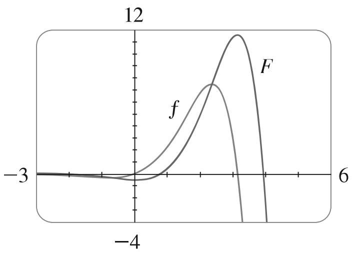



**FIGURE 1**

This can be regarded as an equation to be solved for the unknown integral. Adding $\int e^x \sin x\,dx$ to both sides, we obtain

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

Dividing by 2 and adding the constant of integration, we get

$$\int e^x \sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C$$ ■

If we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts. Evaluating both sides of Formula 1 between $a$ and $b$, assuming $f'$ and $g'$ are continuous, and using the Fundamental Theorem, we obtain

**6** $$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

**EXAMPLE 5** Calculate $\int_0^1 \tan^{-1}x\,dx$.

**SOLUTION** Let

$$u = \tan^{-1}x \qquad dv = dx$$

Then $$du = \frac{dx}{1 + x^2} \qquad v = x$$

So Formula 6 gives

$$\begin{aligned}\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= 1 \cdot \tan^{-1}1 - 0 \cdot \tan^{-1}0 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx\end{aligned}$$

Since $\tan^{-1}x \geqslant 0$ for $x \geqslant 0$, the integral in Example 5 can be interpreted as the area of the region shown in Figure 2.

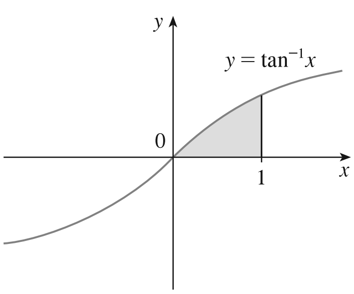



**FIGURE 2**

To evaluate this integral we use the substitution $t = 1 + x^2$ (since $u$ has another meaning in this example). Then $dt = 2x\,dx$, so $x\,dx = \frac{1}{2}\,dt$. When $x = 0$, $t = 1$; when $x = 1$, $t = 2$; so

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int_1^2 \frac{dt}{t} = \tfrac{1}{2}\ln|t|\Big]_1^2 \\ &= \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2\end{aligned}$$

Therefore $$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$ ■

**EXAMPLE 6** Prove the reduction formula

Equation 7 is called a *reduction formula* because the exponent $n$ has been *reduced* to $n - 1$ and $n - 2$.

**7** $$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

**SOLUTION** Let

$$u = \sin^{n-1}x \qquad dv = \sin x\, dx$$

Then $$du = (n-1)\sin^{n-2}x \cos x\, dx \qquad v = -\cos x$$

so integration by parts gives

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\, dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\, dx$$

or $$\int \sin^n x\, dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$$ ■

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\, dx$ in terms of $\int \sin x\, dx$ (if $n$ is odd) or $\int (\sin x)^0\, dx = \int dx$ (if $n$ is even).

## 7.1 EXERCISES

**1–2** Evaluate the integral using integration by parts with the indicated choices of $u$ and $dv$.

**1.** $\int xe^{2x}\, dx; \quad u = x,\ dv = e^{2x}\, dx$

**2.** $\int \sqrt{x}\ln x\, dx; \quad u = \ln x,\ dv = \sqrt{x}\, dx$

**3–36** Evaluate the integral.

**3.** $\int x\cos 5x\, dx$

**4.** $\int ye^{0.2y}\, dy$

**5.** $\int te^{-3t}\, dt$

**6.** $\int (x-1)\sin \pi x\, dx$

**7.** $\int (x^2 + 2x)\cos x\, dx$

**8.** $\int t^2 \sin \beta t\, dt$

**9.** $\int \cos^{-1}x\, dx$

**10.** $\int \ln\sqrt{x}\, dx$

**11.** $\int t^4 \ln t\, dt$

**12.** $\int \tan^{-1} 2y\, dy$

**13.** $\int t\csc^2 t\, dt$

**14.** $\int x\cosh ax\, dx$

**15.** $\int (\ln x)^2\, dx$

**16.** $\int \frac{z}{10^z}\, dz$

**17.** $\int e^{2\theta}\sin 3\theta\, d\theta$

**18.** $\int e^{-\theta}\cos 2\theta\, d\theta$

**19.** $\int z^3 e^z\, dz$

**20.** $\int x\tan^2 x\, dx$

**21.** $\int \frac{xe^{2x}}{(1+2x)^2}\, dx$

**22.** $\int (\arcsin x)^2\, dx$

**23.** $\int_0^{1/2} x\cos \pi x\, dx$

**24.** $\int_0^1 (x^2+1)e^{-x}\, dx$

**25.** $\int_0^2 y\sinh y\, dy$

**26.** $\int_1^2 w^2 \ln w\, dw$

**27.** $\int_1^5 \frac{\ln R}{R^2}\, dR$

**28.** $\int_0^{2\pi} t^2 \sin 2t\, dt$

**29.** $\int_0^{\pi} x\sin x\cos x\, dx$

**30.** $\int_1^{\sqrt{3}} \arctan(1/x)\, dx$

**31.** $\displaystyle\int_1^5 \frac{M}{e^M}\,dM$

**32.** $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$

**33.** $\displaystyle\int_0^{\pi/3} \sin x \ln(\cos x)\,dx$

**34.** $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4 + r^2}}\,dr$

**35.** $\displaystyle\int_1^2 x^4(\ln x)^2\,dx$

**36.** $\displaystyle\int_0^t e^s \sin(t - s)\,ds$

**37–42** First make a substitution and then use integration by parts to evaluate the integral.

**37.** $\displaystyle\int e^{\sqrt{x}}\,dx$

**38.** $\displaystyle\int \cos(\ln x)\,dx$

**39.** $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

**40.** $\displaystyle\int_0^{\pi} e^{\cos t} \sin 2t\,dt$

**41.** $\displaystyle\int x \ln(1 + x)\,dx$

**42.** $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx$

**43–46** Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

**43.** $\displaystyle\int xe^{-2x}\,dx$

**44.** $\displaystyle\int x^{3/2} \ln x\,dx$

**45.** $\displaystyle\int x^3\sqrt{1 + x^2}\,dx$

**46.** $\displaystyle\int x^2 \sin 2x\,dx$

**47.** (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

**48.** (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

**49.** (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.
(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n + 1)}$$

**50.** Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}\,\frac{\pi}{2}$$

**51–54** Use integration by parts to prove the reduction formula.

**51.** $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

**52.** $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

**53.** $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n \neq 1)$

**54.** $\displaystyle\int \sec^n x\,dx = \frac{\tan x \sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n \neq 1)$

**55.** Use Exercise 51 to find $\int (\ln x)^3\,dx$.

**56.** Use Exercise 52 to find $\int x^4 e^x\,dx$.

**57–58** Find the area of the region bounded by the given curves.

**57.** $y = x^2 \ln x, \quad y = 4 \ln x$

**58.** $y = x^2 e^{-x}, \quad y = xe^{-x}$

**59–60** Use a graph to find approximate $x$-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

**59.** $y = \arcsin\left(\frac{1}{2}x\right), \quad y = 2 - x^2$

**60.** $y = x\ln(x + 1), \quad y = 3x - x^2$

**61–64** Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves about the given axis.

**61.** $y = \cos(\pi x/2),\ y = 0,\ 0 \leqslant x \leqslant 1$; about the $y$-axis

**62.** $y = e^x,\ y = e^{-x},\ x = 1$; about the $y$-axis

**63.** $y = e^{-x},\ y = 0,\ x = -1,\ x = 0$; about $x = 1$

**64.** $y = e^x,\ x = 0,\ y = 3$; about the $x$-axis

**65.** Calculate the volume generated by rotating the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 2$ about each axis.
(a) The $y$-axis
(b) The $x$-axis

**66.** Calculate the average value of $f(x) = x\sec^2 x$ on the interval $[0, \pi/4]$.

**67.** The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right)dt$ was discussed in Example 5.3.3 and is used extensively in the theory of optics. Find $\int S(x)\,dx$. [Your answer will involve $S(x)$.]

**68.** A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is $m$, the fuel is consumed at rate $r$, and the exhaust gases are ejected with constant velocity $v_e$ (relative to the rocket). A model for the velocity of the rocket at time $t$ is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where $g$ is the acceleration due to gravity and $t$ is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

**69.** A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after $t$ seconds. How far will it travel during the first $t$ seconds?

**70.** If $f(0) = g(0) = 0$ and $f''$ and $g''$ are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

**71.** Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and $f''$ is continuous. Find the value of $\int_1^4 x f''(x)\,dx$.

**72.** (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If $f$ and $g$ are inverse functions and $f'$ is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]

(c) In the case where $f$ and $g$ are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

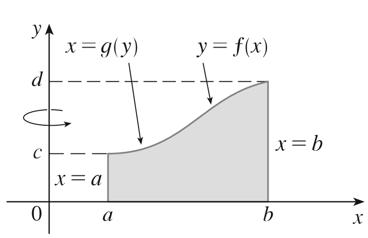
(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

**73.** We arrived at Formula 6.3.2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where $f$ is one-to-one and therefore has an inverse function $g$. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$




**74.** Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.

(b) Use Exercise 50 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.

(d) Use part (c) and Exercises 49 and 50 to show that

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots \cdot \frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot \cdots$$

and is called the *Wallis product*.

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.