10. **(Annihilator)** Let B be a subset of the dual space X' of a normed space X. The *annihilator* ^aB of B is defined to be

$$^aB = \{x \in X \mid f(x) = 0 \text{ for all } f \in B\}.$$

Show that in Prob. 9,

$$\mathcal{R}(T) \subset {}^a\mathcal{N}(T^\times).$$

What does this mean with respect to the task of solving an equation $Tx = y$?

4.6 Reflexive Spaces

Algebraic reflexivity of *vector* spaces was discussed in Sec. 2.8. *Reflexivity* of *normed* spaces will be the topic of the present section. But let us first recall what we did in Sec. 2.8. We remember that a vector space X is said to be *algebraically reflexive* if the canonical mapping $C: X \longrightarrow X^{**}$ is surjective. Here $X^{**} = (X^*)^*$ is the second algebraic dual space of X and the mapping C is defined by $x \longmapsto g_x$ where

(1) $$g_x(f) = f(x) \qquad (f \in X^* \text{ variable});$$

that is, for any $x \in X$ the image is the linear functional g_x defined by (1). If X is finite dimensional, then X is algebraically reflexive. This was shown in Theorem 2.9-3.

Let us now turn to our actual task. We consider a normed space X, its dual space X' as defined in 2.10-3 and, moreover, the dual space $(X')'$ of X'. This space is denoted by X'' and is called the **second dual space** of X (or *bidual space* of X).

We define a functional g_x on X' by choosing a fixed $x \in X$ and setting

(2) $$g_x(f) = f(x) \qquad (f \in X' \text{ variable}).$$

This looks like (1), but note that now f is bounded. And g_x turns out to be bounded, too, since we have the basic

4.6-1 Lemma (Norm of g_x). *For every fixed x in a normed space the functional g_x defined by (2) is a bounded linear functional on X' that $g_x \in X''$, and has the norm*

$$\|g_x\| = \|x\|. \tag{3}$$

Proof. Linearity of g_x is known from Sec. 2.8, and (3) follo from (2) and Corollary 4.3-4:

$$\|g_x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|g_x(f)|}{\|f\|} = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} = \|x\|. \tag{4}$$

To every $x \in X$ there corresponds a unique bounded linear fu tional $g_x \in X''$ given by (2). This defines a mapping

$$\begin{aligned} C\colon X &\longrightarrow X'' \\ x &\longmapsto g_x. \end{aligned} \tag{5}$$

C is called the **canonical mapping** of X into X''. We show that C linear and injective and preserves the norm. This can be expresse terms of an isomorphism of normed spaces as defined in Sec. 2.1

4.6-2 Lemma (Canonical mapping). *The canonical mapping C g by (5) is an isomorphism of the normed space X onto the normed s $\mathcal{R}(C)$, the range of C.*

Proof. Linearity of C is seen as in Sec. 2.8 because

$$g_{\alpha x+\beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f).$$

In particular, $g_x - g_y = g_{x-y}$. Hence by (3) we obtain

$$\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|.$$

This shows that C is isometric; it preserves the norm. Isometry imp injectivity. We can also see this directly from our formula. Indee $x \neq y$, then $g_x \neq g_y$ by axiom (N2) in Sec. 2.2. Hence C is bije regarded as a mapping onto its range. ∎

X is said to be **embeddable** in a normed space Z if X is isomorphic with a subspace of Z. This is similar to Sec. 2.8, but note that here we are dealing with isomorphisms of normed spaces, that is, vector space isomorphisms which preserve norm (cf. Sec. 2.10). Lemma 4.6-2 shows that X is embeddable in X'', and C is also called the *canonical embedding* of X into X''.

In general, C will not be surjective, so that the range $\mathscr{R}(C)$ will be a proper subspace of X''. The surjective case when $\mathscr{R}(C)$ is all of X'' is important enough to give it a name:

4.6-3 Definition (Reflexivity). A normed space X is said to be *reflexive* if

$$\mathscr{R}(C) = X''$$

where $C: X \longrightarrow X''$ is the canonical mapping given by (5) and (2). ∎

This concept was introduced by H. Hahn (1927) and called "reflexivity" by E. R. Lorch (1939). Hahn recognized the importance of reflexivity in his study of linear equations in normed spaces which was motivated by integral equations and also contains the Hahn-Banach theorem as well as the earliest investigation of dual spaces.

If X is reflexive, it is isomorphic (hence isometric) with X'', by Lemma 4.6-2. It is interesting that the converse does not generally hold, as R. C. James (1950, 1951) has shown.

Furthermore, completeness does not imply reflexivity, but conversely we have

4.6-4 Theorem (Completeness). *If a normed space X is reflexive, it is complete (hence a Banach space).*

Proof. Since X'' is the dual space of X', it is complete by Theorem 2.10-4. Reflexivity of X means that $\mathscr{R}(C) = X''$. Completeness of X now follows from that of X'' by Lemma 4.6-2. ∎

R^n is reflexive. This follows directly from 2.10-5. It is typical of any finite dimensional normed space X. Indeed, if $\dim X < \infty$, then every linear functional on X is bounded (cf. 2.7-8), so that $X' = X^*$ and algebraic reflexivity of X (cf. 2.9-3) thus implies

4.6-5 Theorem (Finite dimension). *Every finite dimensional normed space is reflexive.*