spaces over the same field becomes a vector space if we define the two algebraic operations by

$$(x_1, x_2)+(y_1, y_2)=(x_1+y_1, x_2+y_2),$$

$$\alpha(x_1, x_2)=(\alpha x_1, \alpha x_2).$$

14. (Quotient space, codimension) Let Y be a subspace of a vector space X. The *coset* of an element $x \in X$ with respect to Y is denoted by $x+Y$ and is defined to be the set (see Fig. 12)

$$x+Y=\{v \mid v=x+y, y \in Y\}.$$

Show that the distinct cosets form a partition of X. Show that under algebraic operations defined by (see Figs. 13, 14)

$$(w+Y)+(x+Y)=(w+x)+Y$$

$$\alpha(x+Y)=\alpha x+Y$$

these cosets constitute the elements of a vector space. This space is called the *quotient space* (or sometimes *factor space*) *of* X *by* Y (or *modulo* Y) and is denoted by X/Y. Its dimension is called the *codimension* of Y and is denoted by codim Y, that is,

$$\operatorname{codim} Y=\dim (X/Y).$$

15. Let $X=\mathbf{R}^3$ and $Y=\{(\xi_1, 0, 0) \mid \xi_1 \in \mathbf{R}\}$. Find X/Y, X/X, $X/\{0\}$.

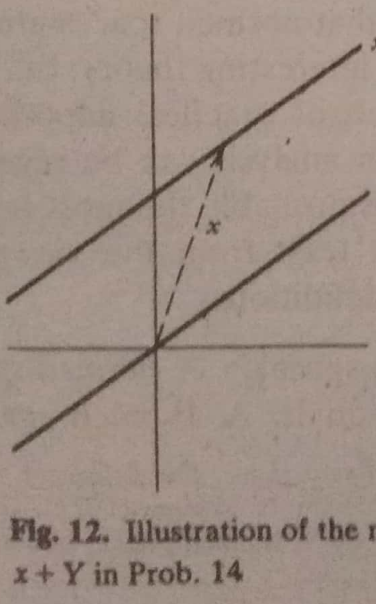

Fig. 12. Illustration of the notation $x+Y$ in Prob. 14

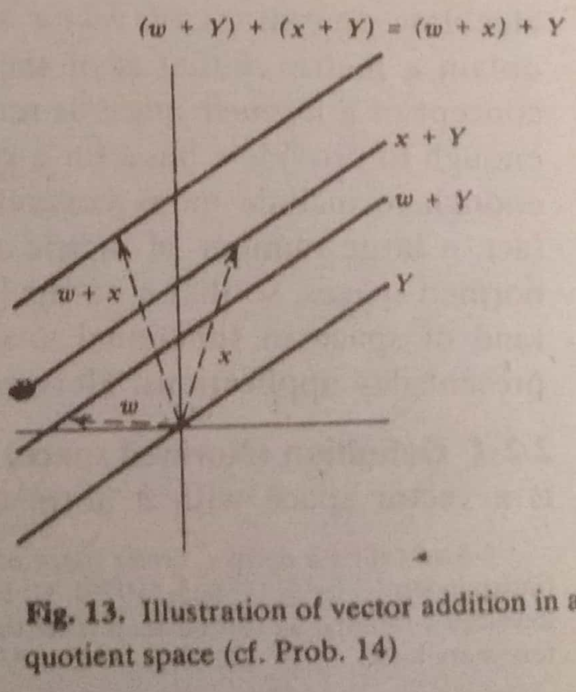

Fig. 13. Illustration of vector addition in a quotient space (cf. Prob. 14)

7. **(Absolute convergence)** Show that convergence of $\|y_1\|+\|y_2\|+\|y_3\|+\cdots$ may not imply convergence of $y_1+y_2+y_3+\cdots$. *Hint.* Consider Y in Prob. 3 and (y_n), where $y_n=(\eta_j^{(n)})$, $\eta_n^{(n)}=1/n^2$, $\eta_j^{(n)}=0$ for all $j\neq n$.

8. If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

9. Show that in a Banach space, an absolutely convergent series is convergent.

10. **(Schauder basis)** Show that if a normed space has a Schauder basis, it is separable.

11. Show that (e_n), where $e_n=(\delta_{nj})$, is a Schauder basis for l^p, where $1\leq p<+\infty$.

12. **(Seminorm)** A *seminorm* on a vector space X is a mapping $p\colon X\longrightarrow \mathbf{R}$ satisfying (N1), (N3), (N4) in Sec. 2.2. (Some authors call this a *pseudonorm.*) Show that

$$p(0)=0,$$

$$|p(y)-p(x)|\leq p(y-x).$$

(Hence if $p(x)=0$ implies $x=0$, then p is a norm.)

13. Show that in Prob. 12, the elements $x\in X$ such that $p(x)=0$ form a subspace N of X and a norm on X/N (cf. Prob. 14, Sec. 2.1) is defined by $\|\hat{x}\|_0=p(x)$, where $x\in\hat{x}$ and $\hat{x}\in X/N$.

14. **(Quotient space)** Let Y be a closed subspace of a normed space $(X,\|\cdot\|)$. Show that a norm $\|\cdot\|_0$ on X/Y (cf. Prob. 14, Sec. 2.1) is defined by

$$\|\hat{x}\|_0=\inf_{x\in\hat{x}}\|x\|$$

where $\hat{x}\in X/Y$, that is, $\hat{x}$ is any coset of Y.

15. **(Product of normed spaces)** If $(X_1,\|\cdot\|_1)$ and $(X_2,\|\cdot\|_2)$ are normed spaces, show that the product vector space $X=X_1\times X_2$ (cf. Prob. 13, Sec. 2.1) becomes a normed space if we define

$$\|x\|=\max\,(\|x_1\|_1,\|x_2\|_2) \qquad [x=(x_1,x_2)].$$