THEOREM:—

Every finite dimensional subspace Y of a normed space $(X, \|\cdot\|)$ is complete. In particular every finite dimensional space is complete.

PROOF:— Let $\dim(Y) = n$ and let $(e_1, \dots, e_n)$ be the basis for Y.

Let $(y_n)_1^\infty$ be the cauchy seq in Y. Then $\forall\ \varepsilon > 0$ $\exists\ N \in \mathbb{N}$ s.t

$$\|y_m - y_r\| < \varepsilon \quad \forall\ m, r > N$$

$$\Rightarrow \quad \varepsilon > \|y_m - y_r\| = \|(\alpha_1^{(m)} e_1 + \dots + \alpha_n^{(m)} e_n) - (\alpha_1^{(r)} e_1 + \dots + \alpha_n^{(r)} e_n)\|$$

$$= \|(\alpha_1^{(m)} - \alpha_1^{(r)}) e_1 + \dots + (\alpha_n^{(m)} - \alpha_n^{(r)}) e_n\|$$

$$\geq c \sum_1^\infty |\alpha_i^{(m)} - \alpha_i^{(r)}|$$

(by previous lemma where $c > 0$)

$\Rightarrow \quad \sum_1^\infty |\alpha_i^{(m)} - \alpha_i^{(r)}| < \varepsilon/c \quad \forall\ m, r > N$

$\Rightarrow \quad \forall$ fixid i $(1 \leq i \leq n)$ we have

$$|\alpha_i^{(m)} - \alpha_i^{(r)}| < \varepsilon \quad \forall\ m, r > N$$

$\Rightarrow \quad \forall$ fixed i $(1 \leq i \leq n)$, $(\alpha_i^{(m)})_{m=1}^\infty$ be a Cauchy seq in $\mathbb{R}$ or $\mathbb{C}$. since, $\mathbb{R}/\mathbb{C}$ are complete $\exists\ \alpha_i \in \mathbb{R}$ (or $\mathbb{C}$) s.t $\alpha_i^{(m)} \longrightarrow \alpha_i$ as $m \longrightarrow \infty$

$\Rightarrow \quad y_m = \alpha_1^{(m)} e_1 + \dots + \alpha_n^{(m)} e_n \longrightarrow \alpha_1 e_1 + \dots + \alpha_n e_n = y \in Y$ as $m \longrightarrow \infty$ i.e $\lim_{m \to \infty} y_m = y$

Now, we check that whether this convergence is under norm or not.

For this, consider

$$\|y_m - y\| = \|(\alpha_1^{(m)} e_1 + \dots + \alpha_n^{(m)} e_n) - (\alpha_1 e_1 + \dots + \alpha_n e_n)\|$$

$$= \|(\alpha_1^{(m)} - \alpha_1) e_1 + \dots + (\alpha_n^{(m)} - \alpha_n) e_n\|$$

$$\leq K \sum_1 |\alpha_i^{(m)} - \alpha_i| \quad ; \quad K = \max\|e_i\|, (1 \leq i \leq n)$$

$$\longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty \quad (\because \alpha_i^{(m)} \longrightarrow \alpha_i \text{ as } m \longrightarrow \infty)$$

$\Rightarrow \quad y_m \xrightarrow{\|\cdot\|} y$ as $m \longrightarrow \infty$

$\Rightarrow \quad Y$ is complete.

THEOREM:—

Every finite dimensional subspace Y of a normed space $(X, \|\cdot\|)$ is closed.

NOTE:— In case of infinite dimensional subspace above result need not be true.

EXP # Let $X = c[a,b]$ with $\|x\| = \max_{t\in[a,b]} |x(t)|$. Then $(c[a,b], \|\cdot\|)$ is complete.

SOL:— Let $Y = \{1, t, t^2, \dots\}$ be the set of all polynomials.

Let $(y_n)_1^\infty$ be a seq in Y. Then, $y_n = 1+t+\frac{t^2}{2!}+\dots+\frac{t^n}{n!}$

$\Rightarrow \lim_{n\to\infty} y_n = 1+t+\frac{t^2}{2!}+\dots+\frac{t^n}{n!}+\dots = e^t \notin Y.$

$\Rightarrow$ Y is not closed and hence not complete.

DEF:— A norm $\|\cdot\|$ on a vector space X is said to be equivalent to norm $\|\cdot\|_0$ on X if $\exists$ +ve numbers α & β s.t $\forall$ $x \in X$ we have

$$\alpha \|x\|_0 \le \|x\| \le \beta \|x\|_0$$

* **EXP #** Let $X = \mathbb{R}^2$. Define two norms on X by $\|x\|_1 = |\xi_1| + |\xi_2|$; $x = (\xi_1, \xi_2)$

$\|x\|_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2} = \left(\sum_1^2 |\xi_i|^2\right)^{1/2}$; $x = (\xi_1, \xi_2)$

Show that $\|\cdot\|_1$ & $\|\cdot\|_2$ are equivalent.

SOL:—
$$\|x\|_1 = |\xi_1| + |\xi_2|$$
$$= \sum_1^2 |\xi_i|$$
$$= \sum_1^2 1\cdot|\xi_i|$$
$$\underset{\substack{H\cdot I \\ \text{For } p,q=2}}{\le} \left(\sum_1^2 (1)^2\right)^{1/2} \cdot \left(\sum_1^2 |\xi_i|^2\right)^{1/2} = \sqrt{2}\,\|x\|_2$$

i.e $\|x\|_1 \leq \sqrt{2}\,\|x\|_2$

$\Rightarrow \frac{1}{\sqrt{2}}\|x\|_1 \leq \|x\|_2$ ——— ①

Now, $\|x\|_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2}$

$\leq |\xi_1| + |\xi_2|$

$\|x\|_2 \leq 1\cdot\|x\|_1$ ——— ②

$\because (|\xi_1| + |\xi_2|)^2$
$= |\xi_1|^2 + |\xi_2|^2 + 2|\xi_1||\xi_2|$
$\geq |\xi_1|^2 + |\xi_2|^2$
$\Rightarrow |\xi_1| + |\xi_2| \geq \sqrt{|\xi_1|^2+|\xi_2|^2}$

From ① & ②

$\frac{1}{\sqrt{2}}\|x\|_1 \leq \|x\|_2 \leq 1\cdot\|x\|_1$

Hence, $\|x\|_1$ & $\|x\|_2$ norms are equivalent.

THEOREM:— On a finite dimensional normed space X any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

PROOF:— Let $\dim X = n$ and let $\{e_1, \ldots, e_n\}$ be the basis for X. Then each $x \in X$ be unique representation of the form

$x = \alpha_1 e_1 + \cdots + \alpha_n e_n$

so that $\|x\| = \|\sum_{i=1}^{n} \alpha_i e_i\| \geq c \sum_{i=1}^{n} |\alpha_i|$

(by previous lemma where $c > 0$)

$\Rightarrow \sum_{i=1}^{n} |\alpha_i| \leq \frac{1}{c}\|x\|$ ——— ①

Now, $\|x\|_0 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_0$

$\leq k \sum_{i=1}^{n} |\alpha_i|$ where $k = \max_{1\leq i\leq n} \|e_i\|_0$

$= \frac{k}{c}\|x\|$

i.e $\|x\|_0 \leq \beta\|x\|$ ——— (a) where $\beta = k/c$

Interchanging the role of norms $\|x\|, \|x\|_0$ in above discussion we get

$\|x\| \leq d\,\|x\|_0$ ——— (b)

Now, (a) & (b) $\Rightarrow$

$$\frac{1}{\beta}\|x\|_0 \le \|x\| \le d\|x\|_0$$

Hence, $\|x\|$, $\|x\|_0$ are equivalent.

REMARK:— In case of finite dimensional space we need not worry about the norm because all the norms are equivalent.

DEF:— A metric space X is said to be compact if every sequency in X has convergent subsequency.

A subset M of X is said to be compact if M itself is compact in subspace of X i.e every sequency in M has convergent subsequency.

REMARK:— There are three ways of defining concept of compactness in a general topological space.

1. Every open cover has a finite subcover.
2. Every convergent cover has finite subcover.
3. Every set has convergent subsequency.

LEMMA:— A compact subset M of a metric space (X,d) (finite or infinite) dimensional) is closed and bounded.

PROOF:— (i) To prove that M is closed.

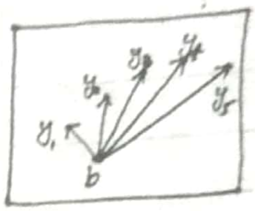
obvious $M \subseteq \bar{M}$.

Let $x \in \bar{M}$ $\exists$ a seq (x_n) in M s.t $x_n \to x$.

Since, M is compact, $(x_n)_1^\infty$ has convergent subsequence. So that $x \in M$. Then, $\bar{M} \subseteq M$. So that $\bar{M} = M$ i.e M is closed.

(ii) To prove that M is bounded.

Suppose that M is not bounded. $\exists$ a seq $(y_n)_1^\infty$ in M s.t $d(y_n, b) > n$ where b is the fixed element of M. Then, $(y_n)_1^\infty$ cannot have convergent subseq. which is contradiction to fact that M is compact. So, M is bounded.



REMARK:— Converse of above theorem is not true in general.

EXP # Consider $l^2 = \{x = (\xi_i)_1^\infty \mid \sum_1^\infty |\xi_i|^2 < \infty\}$ with $\|x\| = (\sum_1^\infty |\xi_i|^2)^{1/2}$.

SOL:— Consider $M = \{e_1 = (1,0,0), e_2 = (0,1,0), \dots\}$

Then, $\delta(M) = \sup_{x,y \in M} \|x-y\| = \sqrt{2} < \infty$

$\Rightarrow M$ is bounded. M is closed because M is discrete or pt set. No limit pts of M exist so, we can safely assume that M contains all its limit pt. M is not compact because seq $(e_i)_1^\infty$ has subseq $e_1, e_{50}, e_{90}, \dots \not\to$ lim pt. which is contradiction. So, M is not closed.

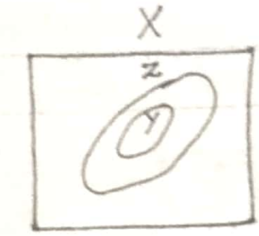
THEOREM:— In a finite dimensional normed space $(X, \|\cdot\|)$ any subset $M \subset X$ is compact iff M is closed and bounded.

PROOF:— If M is compact then it is and bounded by previous theorem.

CONVERSELY:— suppose that M is closed and bounded. we have to prove that M is compact.
Since, X is finite dimensional therefore $\dim X = n$ and let $\{e_1, \ldots, e_n\}$ be the basis for X. Let $(x_m)_1^\infty$ be an arbitrary seq in M. Then, $x_m = \alpha_1^{(m)} e_1 + \cdots + \alpha_n^{(m)} e_n \quad \forall\ m \in \mathbb{N}$
Since, M is bounded. So, $(x_m)_1^\infty$ is also bounded.
Then, $\|x_m\| \leq K$ where $K > 0$
$\Rightarrow \quad K \geq \|x_m\| = \|\alpha_1^{(m)} e_n + \cdots + \alpha_n^{(m)} e_n\| \geq c \sum_1^n |\alpha_i^{(m)}|$
(by previous lemma where $c > 0$)
$\Rightarrow \sum^n |\alpha_i^{(m)}| \leq K/c$
$\Rightarrow \forall$ fixed i $(1 \leq i \leq n)$, $\sum_1^n |\alpha_i^{(m)}| \leq K/c$. Then, $\forall\ i\ (1 \leq i \leq n)$, $(\alpha_i^{(m)})_{m=1}^\infty$ is bounded seq.
By Bolzano Weirstrass theorem $(\alpha_i^{(m)})$ has convergent subsequence $(\gamma_i^{(m)})_{m=1}^\infty$ which converges to γ_i (say).
Then, $(x_m)_1^\infty$ has convergent subseq $(z_m)_1^\infty$ which converges to $\gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_n e_n = z$.
Since, M is closed. so that $z \in M$. Hence, (x_m) has convergent subsequence. This proves that M is compact.

F. RIEZ LEMMA:— Let Y & Z be a subspace of a normed space X (of any dimension) and suppose that Y is closed and is proper subset of Z. Then for every real number $\theta \in (0,1)$ $\exists$ a $z \in Z$ s.t $\|z\| = 1$ and $\|z - y\| \geq \theta \quad \forall\ y \in Y$.

PROOF:— choose $v \in Z/Y$ and let a be the distance of v from Y. Then,



$$a = \inf_{y \in Y} \|v-y\|$$

clearly $a > 0$. Choose $\theta \in (0,1)$ $\exists$ a $y_0 \in Y$ s.t

$$a \leq \|v-y_0\| \leq a/\theta \quad \text{———} \quad ①$$

Note that $a/\theta > a$ because $0 < \theta < 1$. Let $z = c(v-y_0)$ where $c = \dfrac{1}{\|v-y_0\|}$. Then,

$$\|z\| = |c|\,\|v-y_0\| = \frac{1}{\cancel{\|v-y_0\|}} \cdot \cancel{\|v-y_0\|} = 1$$

$$\Rightarrow \|z\| = 1$$

$$\|z-y\| = \|c(v-y_0) - y\| = c\|v-y_0 - \tfrac{1}{c}y\|$$

$$= c\|v-(\underbrace{y_0 + \tfrac{1}{c}y}_{\in Y})\| \quad (\because Y \text{ is subspace})$$

$$\geq ca \quad (\text{by def of inf})$$

$$= \frac{1}{\|v-y_0\|} \cdot a \geq \frac{\theta}{\cancel{a}} \cdot \cancel{a}$$

$$\Rightarrow \|z-y\| \geq \theta$$

DEF:— Let X & Y be v-spaces then an operator $T : X \longrightarrow Y$ is called linear if

fn $\leftarrow f : A \longrightarrow B$ (ordinary set)

operator $\leftarrow T : X \longrightarrow Y$ (v-space)

fnctional $\leftarrow f : X \longrightarrow \mathbb{R}/\mathbb{C}$

(i) $\forall\ x, y \in X$, $T(x+y) = Tx + Ty$

(ii) $\forall$ scalars α & $\forall\ x \in X$ $T(\alpha x) = \alpha Tx$

Equivalently, $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$

$$T(\alpha x + \beta y) = T(\alpha x) + T(\beta y)$$

$$= \alpha Tx + \beta Ty$$

DEF:— Let $T: X \to Y$ be an operator. Then, the null space of T denoted by $N(T)$ is defined as $N(T) = \{x \in X \mid Tx = 0\}$.

NOTE:— If we choose $\alpha = 0$ in $T(\alpha x) = \alpha Tx$ we get $T(0) = 0$.

EXP # 1 The identity operator $I: X \to X$ defined by $Ix = x$ $\forall x \in X$ is linear because
$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha Ix + \beta Iy$

EXP # 2 The zero operator $0: X \to Y$ defined by $0x = 0$ is linear because
$0(\alpha x + \beta y) = 0 = \alpha 0x + \beta 0y$

EXP # 3 Let X be vector space of all polynomials on $[a,b]$ defined $T: X \to X$ by $Tx' = x'$ $\forall x \in X$. $x' = \frac{dx}{dt}$; $t \in [a,b]$. T is linear because
$T(\alpha x + \beta y) = (\alpha x + \beta y)' = \alpha x' + \beta y' = \alpha Tx' + \beta Ty'$

EXP # 4 Consider $T: C[a,b] \to \mathbb{R}$ defined by $Tx = \int_a^b x(t)\,dt$ $\forall x \in C[a,b]$. Then, T is linear because

$$T(\alpha x + \beta y) = \int_a^b (\alpha(x(t)) + \beta(y(t)))\,dt$$

$$= \alpha \int_a^b x(t)\,dt + \beta \int_a^b y(t)\,dt = \alpha Tx + \beta Ty$$

EXP # 5 Define $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ by $T_1 x = x \times a$ $\forall x \in \mathbb{R}^3$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is fixed vector in $\mathbb{R}^3$. Then,

$$T_1(\alpha x+\beta y) = (\alpha x+\beta y)\times a$$
$$= \alpha(x\times a)+\beta(y\times a)$$
$$= \alpha T_1 x+\beta T_1 y$$

EXP # 6 Define $T_2 : \mathbb{R} \to \mathbb{R}$ by $T_2 x = x\cdot a$ $\forall\, x \in X$. Then, T_2 is linear because

$$T_2(\alpha x+\beta y) = (\alpha x+\beta y)\cdot a = \alpha(x\cdot a)+\beta(y\cdot a)$$
$$= \alpha T_2 x+\beta T_2 y$$

EXP # 7 Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} = y$$

So that metrix $(\alpha_{ij})_{m\times n} = A$ can be used as operator from $\mathbb{R}^n \to \mathbb{R}^m$, i.e $A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $Ax = y$. This operator A is linear because metrix mulplication is linear. If A is complex vector it will define operator from $\mathbb{C}^n \to \mathbb{C}^m$.

THEOREM :— Let T be a linear operator. Then,

(2)

The range of T, $R(T)$ is a vector space.

PROOF :— Let $y_1, y_2 \in R(T)$, then $\exists$ $x_1, x_2 \in D(T)$

s.t $Tx_1 = y_1$ & $Tx_2 = y_2$

consider $\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2$ $\because T$ is linear
$= T(\alpha x_1 + \beta x_2) \in R(T)$

$\Rightarrow R(T)$ is a subspace of X and hence a vector space.

(b)

If $\dim D(T) = n$, then $\dim R(T) \leq n$ i.e $\dim R(T) \leq \dim D(T)$.

PROOF:— let $y_1, \dots, y_{n+1} \in R(T)$, then $\exists$ $x_1, \dots, x_{n+1} \in D(T)$ s.t $Tx_1 = y_1, \dots, Tx_{n+1} = y_{n+1}$

Since, $\dim D(T) = n$. Therefore $x_1, \dots, x_{n+1}$ is linearly dependent. Then,

$\alpha x_1 + \dots + \alpha x_{n+1} = 0 \Rightarrow$ all α_i's $\neq 0$

$\Rightarrow T(\alpha x_1 + \dots + \alpha x_{n+1}) = T(0) = 0$ where all α_i's $\neq 0$

$\Rightarrow \alpha Tx_1 + \dots + \alpha Tx_{n+1} = 0$ where all α_i's $\neq 0$

$\Rightarrow \alpha y_1 + \dots + \alpha y_{n+1} \underset{T \text{ is linear}}{=} 0$ where all α_i's $\neq 0$

$\Rightarrow y_1 + \dots + y_{n+1}$ are linearly dependent.

So, $\dim R(T) \leq n$.

(c)

The null space $N(T)$ is a vector space.

PROOF:— let $x_1, x_2 \in N(T)$, then $Tx_1 = 0 = Tx_2$

consider $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$
$= 0$ $\because T$ is linear

$\Rightarrow \alpha x_1 + \beta x_2 \in N(T)$.

$\Rightarrow N(T)$ is subspace of X and hence vector space.

DEF:— An operator $T: D(T) \longrightarrow Y$ is said to be injective or one-to-one if different pts in the domain have different images i.e $x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$
or equivalently

$$x_1 = x_2 \Longrightarrow Tx_1 = Tx_2$$

in this case $\exists$ a mapping $T^{-1}: R(T) \longrightarrow D(T)$.

NOTE THAT:—

$$T^{-1}Tx = x \qquad \forall\, x \in D(T)$$
$$TT^{-1}x = x \qquad \forall\, x \in R(T)$$

THEOREM:— Let X and Y be vector spaces both real or both complex. Let $T: D(T) \longrightarrow Y$ be a linear operator with $D(T) \subset X$ and $R(T) \subset Y$. Then,

(a)

The inverse $T^{-1}: R(T) \longrightarrow D(T)$ exists iff $Tx = 0 \Rightarrow x = 0$.

PROOF:— suppose $Tx = 0 \Rightarrow x = 0$. we have to prove that $T^{-1}: R(T) \longrightarrow D(T)$ exists. For this, we have to prove that T is 1-1.
Let $x_1, x_2 \in D(T)$ s.t $Tx_1 = Tx_2$

$\Rightarrow Tx_1 - Tx_2 = 0$

$\Rightarrow T(x_1 - x_2) = 0$ $\qquad \because T$ is linear

$\Rightarrow x_1 - x_2 = 0$ $\qquad$ (by assumption)

$\Rightarrow x_1 = x_2$

CONVERSELY:— suppose that $T^{-1}: R(T) \longrightarrow D(T)$ exists. Then, T is 1-1.
Then, $Tx_1 = Tx_2 \Rightarrow x_1 = x_2 \quad \forall\, x_1, x_2 \in D(T)$.

choosing $x_1 = x$ & $x_2 = 0$ we get

$\Rightarrow Tx = T0 \Rightarrow x = 0$

$\Rightarrow Tx = 0 \Rightarrow x = 0$ ($\because$ if T is linear then $T0 = 0$)

(b)

If T^{-1} exists then it is linear.

PROOF:— Let $y_1, y_2 \in R(T)$ then $\exists$ $x_1, x_2 \in D(T)$ s.t $Tx_1 = y_1$ & $Tx_2 = y_2$.

Since, T^{-1} exists then,

$$x_1 = T^{-1}y_1 \quad \& \quad x_2 = T^{-1}y_2$$

Then, $\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2$

$= T(\alpha x_1 + \beta x_2)$ $\because$ T is linear

$\Rightarrow T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}y_1 + \beta T^{-1}y_2$

So, T^{-1} is Linear.

(C)

If $\dim D(T) = n$ and T^{-1} exists then $\dim D(T) = \dim R(T)$.

PROOF:— we have to prove that if T is linear then

$$\dim R(T) \le \dim D(T) \quad \text{—— ①}$$

Since, T^{-1} exists then

$$\dim R(T^{-1}) \le \dim D(T^{-1})$$

$$\Rightarrow \dim D(T) \le \dim R(T) \quad \text{—— ②}$$

From ① & ②

$$\dim R(T) = \dim D(T)$$

DEF:— Let $(X, \|\cdot\|)$ & $(Y, \|\cdot\|)$ be normed spaces and $T: D(T) \subset X \longrightarrow Y$ be a linear operator. T is called bounded if $\exists$ a real number c s.t

$\|Tx\| \le C\|x\|$ —— ① $\forall\, x \in D(T)$

Q: what is the min value of c for which ① holds?

ANS:— ① $\Rightarrow \dfrac{\|Tx\|}{\|x\|} \le C \quad \forall\, x \in D(T)$ s.t $x \neq 0$

$$\Rightarrow \sup_{\substack{x \in D(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \le C \qquad C \ge \sup_{\substack{x \in D(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

Hence, min value of c for which ① holds is $\sup_{\substack{x \in D(T)\\ x \neq 0}} \dfrac{\|Tx\|}{\|x\|}$. This quantity is denoted by $\|T\|$..

Thus, $\|T\| = \sup_{\substack{x \in D(T)\\ x \neq 0}} \dfrac{\|Tx\|}{\|x\|}$.

Here, T is called norm of operator T.

So, ① with c replaced by $\|T\|$ becomes

$\|Tx\| \le \|T\|\,\|x\|$ —— ②

LEMMA:— Let $T : D(T) \subset X \longrightarrow Y$ be a bounded linear operator then,

(a)

☆ An alternate formula for norm of T is

$$\|T\| = \sup_{\substack{x \in D(T)\\ \|x\| = 1}} \|Tx\|.$$

PROOF:— Let $x \in D(T)$ s.t $x \neq 0$ define by $y = \dfrac{1}{\|x\|} x$ then $\|y\| = 1$.

Consider $\|T\| = \sup_{\substack{x \in D(T)\\ x \neq 0}} \dfrac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in D(T)\\ x \neq 0}} \dfrac{1}{\|x\|} \|Tx\|$

$$= \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| \frac{1}{\|x\|} Tx \right\| \quad \because \|\alpha x\| = |\alpha| \|x\|$$

$$= \sup_{\substack{x \in D(T) \\ x \neq 0}} \left\| T\left(\frac{1}{\|x\|} x\right) \right\| \quad \because T \text{ is linear}$$

$$= \sup_{\substack{x \in D(T) \\ \|y\| = 1}} \|Ty\| \quad \because y \text{ is arbitrary}$$

$$= \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|Tx\|$$

(b)

The norm defined by $\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ satisfies all properties of norm.

PROOF:— N_1: $\because \frac{\|Tx\|}{\|x\|} \geq 0$, therefore $\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \geq 0$

N_2: $\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \iff Tx = 0 \quad \forall x \in D(T) \iff T = 0.$

N_3: $\|\alpha T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|\alpha Tx\|}{\|x\|} = |\alpha| \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = |\alpha| \|T\|$

N_4: $\|T_1 + T_2\| = \sup_{\substack{x \in D(T_1+T_2) \\ x \neq 0}} \frac{\|(T_1 + T_2)x\|}{\|x\|} = \sup_{\substack{x \in D(T_1+T_2) \\ x \neq 0}} \frac{\|T_1 x + T_2 x\|}{\|x\|}$

$$\leq \sup_{\substack{x \in D(T_1+T_2) \\ x \neq 0}} \frac{\|T_1 x\| + \|T_2 x\|}{\|x\|}$$

$$= \sup_{\substack{x \in D(T_1) \\ x \neq 0}} \frac{\|T_1 x\|}{\|x\|} + \sup_{\substack{x \in D(T_2) \\ x \neq 0}} \frac{\|T_2 x\|}{\|x\|}$$

$$= \|T_1\| + \|T_2\|$$

$$\Rightarrow \quad \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

Hence, norm defined by $\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ satisfies all properties of norm.

EXP # The identity operator $I: X \to X$ defined by $Ix = x \ \forall \ x \in X$. This operator is bounded.

Since, $\|Ix\| = \|x\| \leq 1\|x\|$

Now, $\|I\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Ix\|}{\|x\|} = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|x\|}{\|x\|} = 1$

EXP # The zero operator $0: X \to Y$ defined by $0x = 0 \ \forall \ x \in X$ is bounded because

$\|0x\| = \|0\| = 0 \leq c\|x\|$

Now, $\|0\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|0x\|}{\|x\|} = 0$

EXP # Let X be the space of all polynomials on $[0,1]$ with norm

$\|x\| = \max_{t \in [0,1]} \|x(t)\|$ $\qquad$ $\|Tx\| \leq C\|x\| \ \forall \ x \in X$

Define $T: X \to X$ by $Tx = x'(t)$.

obviously T is linear (already checked).

we claim that T is not bounded. To see this consider the seq $(x_n)_1^\infty$ in X defined by

$x_n = t^n \quad ; \quad t \in [0,1]$.

Then, $\|x_n\| = \max_{t \in [0,1]} |x_n(t)|$

$= \max_{t \in [0,1]} |t^n| = 1$

Now, $Tx_n = nt^{n-1}$.

Then, $\|Tx_n\| = \max_{t \in [0,1]} |nt^{n-1}|$

$= n \max_{t \in [0,1]} |t^{n-1}| = n$

Now, consider $\frac{\|Tx_n\|}{\|x_n\|} = n$

Since, $n \in \mathbb{N}$ is arbitrary, therefore $\nexists$ a $c>0$ s.t $\|Tx_n\| \leq c\|x_n\|$.

$\Rightarrow$ T is not bounded.

EXP # Define $T: C[0,1] \longrightarrow C[0,1]$ by

$$Tx = y = \int_0^1 K(t,\tau)x(\tau)d\tau$$

where $K(t,\tau)$ is given fn and is bdd on $[0,1]\times[0,1]$

we show that T is bounded.

SOL:— $\because$ $K(t,\tau)$ is bdd $\exists$ k_0 s.t

$$K(t,\tau) \leq k_0$$

consider $\|Tx\| = \|y\| = \max_{t\in[0,1]} |y(t)|$

$$= \max_{t\in[0,1]} \left|\int_0^1 K(t,\tau)x(\tau)d\tau\right|$$

$$\leq \max_{t\in[0,1]} \left[\int_0^1 |K(t,\tau)||x(\tau)|d\tau\right] \qquad \because \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

$$\leq K_0 \max_{t\in[0,1]} \int_0^1 |x(\tau)|d\tau$$

$$= K_0 \int_0^1 \max_{t\in[0,1]} |x(t)|\,dt$$

$$= K_0 \|x\| \int_0^1 dt$$

$\Rightarrow$ $\|Tx\| \leq K_0\|x\|$

$\Rightarrow$ T is bdd.

EXP # Recall that $A = (a_{ij})_{m\times n}$ define a linear operator $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $Ax = y$ where $x = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix}$ & $y = \begin{pmatrix}\eta_1 \\ \vdots \\ \eta_n\end{pmatrix}$.

Then $Ax = y$ gives $\eta_i = \sum_{j=1}^{n} a_{ij}\xi_j$; $i = 1,\dots,m$

Now, we show that A is bdd.

$$\|Ax\|^2 = \|y\|^2$$

$$= \sum_{i=1}^{m} |\eta_i|^2$$

$$= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} \xi_j \right|^2$$

$$\leq \sum_{i=1}^{m} \left[\left(\sum_{j=1}^{n} (a_{ij})^2 \right)^{1/2} \left(\sum_{j=1}^{n} |\xi_j|^2 \right)^{1/2} \right]^2$$

H.I for $p=2=q$

$$= \|x\|^2 \sum_{i=1}^{m} \left[\sum_{j=1}^{n} (a_{ij})^2 \right] \Rightarrow \|Ax\| \leq c\|x\|$$

$\Rightarrow$ A is bdd.

THEOREM:— If a normed space X is finite dimensional. Then every linear operator on X is bdd.

PROOF:— Let $\dim X = n$ and let $\{e_1, \dots, e_n\}$ be the basis for X. Then every $x \in X$ can be expressed as

$$x = \sum_{1}^{n} \alpha_i e_i \quad \text{—— ①}$$

$\Rightarrow Tx = T\left(\sum_{1}^{n} \alpha_i e_i\right) = \sum_{1}^{n} \alpha_i Te_i$ (∵ T is linear)

$\Rightarrow \|Tx\| = \left\|\sum_{1}^{n} \alpha_i Te_i\right\|$ (by triangular

$\leq \sum_{1}^{n} |\alpha_i| \|Te_i\|$ inequality & $\|\alpha x\| \leq |\alpha| \|x\|$)

$\|Tx\| \leq K \sum_{1}^{n} |\alpha_i|$ —— ② where $K = \max_{1 \leq i \leq n} \|Te_i\|$

From ①

$\|x\| = \left\|\sum_{1}^{n} \alpha_i e_i\right\|$

$\geq c \sum_{1}^{n} |\alpha_i|$ (by Lemma)

$\Rightarrow \sum_{1}^{n} |\alpha_i| \leq \frac{1}{c} \|x\|$ —— ③

using ③ in ②

$\|Tx\| \leq \frac{K}{c} \|x\|$

$\Rightarrow \|Tx\| \leq M \|x\|$ where $M = K/c$.

$\Rightarrow$ T is bdd.

DEF:— Let $T: X \to Y$ be any operator (not necessarily linear) then T is continuous at a pt $x_0 \in X$ if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t $\|Tx - Tx_0\| <$ whenever $\|x - x_0\| < \delta$. T is continuous if it is continuous $\forall\ x \in D(T)$.

THEOREM:— Let $T: D(T) \to Y$ be a linear operator where $D(T) \subset X$ and X & Y are normed spaces then

(2)

T is continuous $\Longleftrightarrow$ T is bounded.

PROOF:— If $T = 0$ then statement is trivially true. Let $T \neq 0$, then $\|T\| \neq 0$. Assume that T is bdd we have to show that T is continuous. Let $\varepsilon > 0$ be given. Define $\delta = \dfrac{\varepsilon}{\|T\|}$ s.t $\|x - x_0\| < \delta$.

$$
\begin{aligned}
\text{Then, } \|Tx - Tx_0\| &= \|T(x - x_0)\| \qquad (\because T \text{ is linear}) \\
&\leq \|T\|\,\|x - x_0\| \\
&< \|T\|\,\delta \\
&= \cancel{\|T\|}\,\frac{\varepsilon}{\cancel{\|T\|}} = \varepsilon
\end{aligned}
$$

$\Rightarrow$ T is continuous at $x_0 \in X$. Since, $x_0 \in D(T)$ is arbitrary, therefore T is continuous.

CONVERSELY:— suppose that T is continuous at arbitrary pt $x_0 \in D(T)$. we have to show that T is bdd. Then $\forall\ \varepsilon > 0$ $\exists\ \delta > 0$ s.t $\|Tx - Tx_0\| \leq \varepsilon$ whenever $\|x - x_0\| \leq \delta$

Now, choose $y \neq 0 \in D(T)$ and define $x = x_0 + \frac{\delta}{\|y\|} y$; then $\|x - x_0\| = \delta$

Then, from ① we have

$\|Tx - Tx_0\| < \varepsilon$

$\Rightarrow \|T(x - x_0)\| < \varepsilon$ (∵ T is linear)

$\|T \frac{\delta}{\|y\|} y\| \leq \varepsilon$

$\Rightarrow \|\frac{\delta}{\|y\|} Ty\| \leq \varepsilon$ (∵ T is linear)

$\Rightarrow \frac{\delta}{\|y\|} \|Ty\| \leq \varepsilon$

$\Rightarrow \|Ty\| \leq \frac{\|y\|}{\delta} \varepsilon$

$\Rightarrow \|Ty\| \leq M \|y\|$

$\Rightarrow$ T is bdd.

(b)

If T is continuous at single pt it is continuous at D(T).

PROOF:— Suppose T is continuous at single pt then from (converse of (a)) T is bdd and hence continuous from (a).

COROLLARY:— Let T be bdd linear operator then

(2)

$x_n \rightarrow x \Rightarrow Tx_n \rightarrow Tx$ where $x_n, x \in D(T)$.

PROOF:— Since T is bdd therefore from (a) of above theorem) T is continuous then by theorem $x_n \to x \Rightarrow Tx_n \to Tx$.

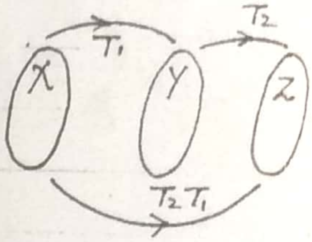
(b)

The null space $N(T)$ is closed i.e $N(T) = \overline{N(T)}$.

PROOF:— we know that $N(T) \subseteq \overline{N(T)}$. Let $x \in \overline{N(T)}$ $\exists$ a seq $(x_n)_1^\infty$ in $N(T)$ s.t $x_n \to x$ then by (a) $Tx_n \to Tx$. Since, $Tx_n = 0$ $\forall\, n \in \mathbb{N}$. ($\because Tx_n \in N(T)$ $\forall\, n \in \mathbb{N}$), therefore $Tx = 0$. So that $x \in N(T)$ and we have $\overline{N(T)} \subseteq N(T)$ i.e $\overline{N(T)} = N(T)$. Hence, $N(T)$ is closed.

DEF:— let $T_1 : X \to Y$ & $T_2 : Y \to Z$ be operators. Then compositions $T_2T_1 : X \to Z$ is defined as

$$T_2T_1(x) = T_2(T_1x).$$



RESULT:— If $T_1 : X \to Y$ & $T_2 : Y \to Z$ are bdd linear operators, then T_2T_1 is also linear & bdd. Moreover $\|T_2T_1\| \le \|T_2\|\|T_1\|$.

PROOF:— T_2, T_1 is linear :—

$$\begin{aligned}
T_2T_1(\alpha x + \beta y) &= T_2(T_1(\alpha x + \beta y)) \\
&= T_2(\alpha T_1 x + \beta T_1 y) && (\because T_1 \text{ is linear}) \\
&= \alpha T_2(T_1x) + \beta T_2(T_1y) && (\because T_2 \text{ is linear}) \\
&= \alpha T_2T_1(x) + \beta T_2T_1(y)
\end{aligned}$$

$\Rightarrow$ T_2T_1 is linear.

T_2, T_1 is bdd :—

$$\begin{aligned}
\|T_2T_1x\| &= \|T_2(T_1x)\| \\
&\le C_2\|T_1x\| && (\because T_2 \text{ is bdd})
\end{aligned}$$

$\leq C_2 C_1 \|x\|$ (∵ T_1 is bdd)

$\Rightarrow \|T_2 T_1 x\| \leq M \|x\|$

$\Rightarrow T_2 T_1$ is bdd

$$\|T_2 T_1\| = \sup_{\substack{x \in D(T_2 T_1) \\ x \neq 0}} \frac{\|T_2 T_1 x\|}{\|x\|} = \sup_{\substack{x \in D(T_2 T_1) \\ x \neq 0}} \frac{\|T_2 (T_1 x)\|}{\|x\|}$$

$$\leq \sup_{\substack{x \in D(T_2 T_1) \\ x \neq 0}} \frac{\|T_2\| \|T_1 x\|}{\|x\|}$$

$$\leq \|T_2\| \sup_{\substack{x \in D(T_1) \\ x \neq 0}} \frac{\|T_1 x\|}{\|x\|}$$

$\Rightarrow \|T_2 T_1\| \leq \|T_2\| \|T_1\|$

Scanned with CamScanner