or

$$\iiint_V \nabla \cdot \mathbf{A}\, dV = \iint_S \mathbf{A} \cdot \hat{\mathbf{n}}\, dS \tag{73}$$

The Divergence theorem gives a passage from volume integral to surface integral. It is a generalization of Green's theorem in the plane and is called Green's theorem in space.

1.22 GREEN'S THEOREMS

1.22.1 Green's First Identity or Theorem

$$\iiint_V [\phi\nabla^2\Psi + (\nabla\phi) \cdot (\nabla\Psi)]\, dV = \iint_S (\phi\nabla\Psi) \cdot d\mathbf{S} \tag{74}$$

This is easily obtained by substituting $\mathbf{A} = \phi\nabla\Psi$ in (73).

1.22.2 Green's Second Identity or Theorem

$$\iiint_V (\phi\nabla^2\Psi - \Psi\nabla^2\phi)\, dV = \iint_S (\phi\nabla\Psi - \Psi\nabla\phi) \cdot d\mathbf{S} \tag{75}$$

This is obtained by interchanging ϕ and Ψ in (74) and subtracting the result from (74). Green's identities are most frequently encountered as transformation formulae in mathematical physics.

Example 47

Verify Stokes' theorem for $\mathbf{A} = (x - y)\,\mathbf{i} + yz^2\,\mathbf{j} - y^2z^2\,\mathbf{k}$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C its boundary.*

The boundary C of S is a circle in xy-plane of radius unity and centre at the origin. Let $x = \cos\theta, y = \sin\theta, \; z = 0, \; 0 \le \theta \le 2\pi$ be the parametric equations of C. Then,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C (x - y)\, dx + yz^2\, dy - y^2z^2\, dz$$

$$= \int_0^{2\pi} (\cos\theta - \sin\theta)(-\sin\theta\, d\theta) = \pi$$

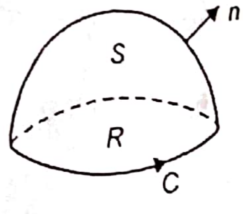

Fig. 1.34

Also,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x - y & yz^2 & -y^2z^2 \end{vmatrix}$$

$$= \mathbf{i}(-2yz^2 - 2yz^2) + \mathbf{j}(0) + \mathbf{k}(1) = \mathbf{k}$$

since $z = 0$.

Then $\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{k} \cdot \mathbf{n}\, dS = \iint_R dx\, dy$

since $\mathbf{n} \cdot \mathbf{k}\, dS = dx\, dy$ and R is the projection of S on the xy-plane. But the area of the circle of unit radius is π. Thus Stokes' theorem is verified.

Example 48

Given $A = 3y\hat{i} + x\hat{j} + 2z\hat{k}$, *find* $\int (\nabla \times A) \cdot \hat{n}\, dS$ *over the hemisphere* $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

First method

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & x & 2z \end{vmatrix} = -2\hat{k}.$$

Since this plane area is in the xy-plane, we have

$$\hat{n} = \hat{k};\ (\nabla \times A) \cdot \hat{n} = -2\hat{k} \cdot \hat{k} = -2$$

so the integral is

$$-2\int ds = -2 \cdot \pi a^2 = -2\pi a^2$$

Second method

Using Stoke's theorem we evaluate $\oint_C A \cdot dr$ around the circle $x^2 + y^2 = a^2$ in the xy-plane

$$\oint_C A \cdot dr = \oint(3y\,\hat{i} + x\,\hat{j} + 2z\,\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint 3y\,dx + x\,dy + 2z\,dz.$$

Put $x = a\cos\theta$, $dx = -a\sin\theta\, d\theta$; $y = a\sin\theta$, $dy = a\cos\theta\, d\theta$;
$z = 0$, with $0 \leq \theta \leq 2\pi$.

$$\oint A \cdot dr = -3a^2 \int_0^{2\pi} \sin^2\theta\, d\theta + a^2 \int_0^{2\pi} \cos^2\theta\, d\theta = -2\pi a^2.$$

Example 49

Use the divergence theorem to show that $\int_S A \cdot dS = \frac{12}{5}\pi R^5$ *where* S *is the sphere of radius* R *and* $A = \hat{i}x^3 + \hat{j}y^3 + \hat{k}z^3$.

$$\iint A \cdot dS = \iiint \nabla \cdot A\, dV$$

But $$\nabla \cdot A = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3R^2,$$

since $x^2 + y^2 + z^2 = R^2$.

$$\int A \cdot dS = \iiint 3R^2\, dV = \int (3R^2)(4\pi R^2\, dR)$$

$$= 12\pi \int R^4\, dR = \frac{12}{5}\pi R^5.$$

Example 50

Evaluate $\oint$ **A . n** dS *over the closed surface of an open cylinder, of height* h *and radius* R.

By the divergence theorem, this is equal to $\int \nabla . A dV$ over the volume of the cylinder

$$\nabla . A = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

$$\oint A . n\, dS = \int \nabla . A\, dV = \int 3\, dV = 3V$$

Surface of cylinder — Volume of cylinder

= 3 times volume of cylinder = $3\pi R^2 h$.

Example 51

Show that the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *is irrotational.*

Solution

To show $\nabla \times \mathbf{r} = 0$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)$$

$$- \hat{\mathbf{j}}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0 + 0 + 0 = 0$$

Example 52

If $\phi = 4x^3 + 3yz^2 - z^3$, *find* $\nabla^2\phi$ *at* $(1, -1, -1)$.

Solution

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= 24x + 0 + (6y - 6z)$$

$$= [(24)(1) + (6)(-1) - (6)(-1)]$$

$$= 24$$

Example 53

Show that $\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B}).\mathbf{C}$

Solution

$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = - \begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \mathbf{C}.(\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \times \mathbf{B}).\mathbf{C}$$

Example 54

Find $\nabla\left(\frac{1}{r^2}\right)$, *where* **r** *is position vector.*

Solution

$$\nabla\left(\frac{1}{r^2}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1}$$

$$= \frac{-(2xi + 2yj + 2zk)}{(x^2 + y^2 + z^2)^2} = \frac{-2(xi + yj + zk)}{r^4}$$

$$= \frac{-2\mathbf{r}}{r^4}$$

Example 55

Prove that $\nabla \cdot \mathbf{r} = 3$ *(***r*** is a position vector). [OU March'97] [OU March'99]*

Solution

$$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

[Refer Example 29]

Example 56

For scalar function $\phi(x, y, z)$ *and a vector* $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ *show that* $\nabla \times (\phi \mathbf{A}) = \phi(\nabla \times \mathbf{A}) + \nabla\phi \times \mathbf{A}$.

Solution

$$\nabla \times (\phi \mathbf{A}) = \nabla \times (\phi A_1\mathbf{i} + \phi A_2\mathbf{j} + \phi A_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]\mathbf{k}$$

$$= \left[\phi\frac{\partial A_3}{\partial y} + \frac{\partial \phi}{\partial y}A_3 - \phi\frac{\partial A_2}{\partial z} - \frac{\partial \phi}{\partial z}A_2\right]\mathbf{i}$$

$$+ \left[\phi\frac{\partial A_1}{\partial z} + \frac{\partial \phi}{\partial z}A_1 - \phi\frac{\partial A_3}{\partial x} - \frac{\partial \phi A_3}{\partial x}\right]\mathbf{j} + \left[\phi\frac{\partial A_2}{\partial x} + \frac{\partial \phi A_2}{\partial x} - \phi\frac{\partial A_1}{\partial y} - \frac{\partial \phi}{\partial y}A_1\right]\mathbf{k}$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$+ \left[\left(\frac{\partial \phi}{\partial y}A_3 - \frac{\partial \phi}{\partial z}A_2\right)\mathbf{i} + \left(\frac{\partial \phi}{\partial z}A_1 - \frac{\partial \phi}{\partial x}A_3\right)\mathbf{j} + \left(\frac{\partial \phi}{\partial x}A_2 - \frac{\partial \phi}{\partial y}A_1\right)\mathbf{k}\right]$$

$$= \phi(\nabla \times A) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\nabla \times A) + (\nabla\phi) \times A$$

Example 57

If ϕ is a scalar quantity show that curl (grad ϕ) = 0.

Solution

$$\nabla \times (\nabla\phi) = \nabla \times \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right).$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]\mathbf{k}$$

$$= \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right]\mathbf{i} - \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right]\mathbf{j} + \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right]\mathbf{k}$$

$$= 0$$

Example 58

Find $\nabla\phi$ *where* $\phi = \log|\mathbf{r}|$

Solution

$|\mathbf{r}| = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\phi = \log|\mathbf{r}| = \frac{1}{2}\log(x^2 + y^2 + z^2)$

$$\nabla\phi = \frac{1}{2}\nabla\log(x^2 + y^2 - z^2)$$

$$= \frac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\log(x^2 + y^2 + z^2) + \mathbf{j}\frac{\partial}{\partial y}\log(x^2 + y^2 + z^2) + \mathbf{k}\frac{\partial}{\partial z}\log(x^2 + y^2 + z^2)\right\}$$

$$= \frac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2 + y^2 + z^2} + \mathbf{j}\frac{2y}{x^2 + y^2 + z^2} + \mathbf{k}\frac{2z}{x^2 + y^2 + z^2}\right\}$$

$$= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^2} = \frac{\mathbf{r}}{r^2}$$

Example 59

Prove that $\nabla.(\mathbf{A} + \mathbf{B}) = \nabla.\mathbf{A} + \nabla.\mathbf{B}$.

Solution

Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

Then,

$$\nabla.(\mathbf{A} + \mathbf{B}) = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right).\left[(A_1 + B_1)\mathbf{i} + (A_2 + B_2)\mathbf{j} + (A_3 + B_3)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}(A_1 + B_1) + \frac{\partial}{\partial y}(A_2 + B_2) + \frac{\partial}{\partial z}(A_3 + B_3)$$

$$= \frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial B_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial B_3}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).(B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k})$$

$$= \nabla.\mathbf{A} + \nabla.\mathbf{B}$$

Example 60

Show that div $(\phi\mathbf{A}) = \phi\, div\, \mathbf{A} + \mathbf{A}.\text{grad}\,\phi$.

Solution

$$\text{div}\,(\phi A) = \nabla(\phi \mathbf{A}) = \nabla.(\phi A_1 \mathbf{i} + \phi A_2 \mathbf{j} + \phi A_3 \mathbf{k})$$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial \phi}{\partial x} A_1 + \phi \frac{\partial A_1}{\partial x} + \frac{\partial \phi}{\partial y} A_2 + \phi \frac{\partial A_2}{\partial y} + \frac{\partial \phi}{\partial z} A_3 + \phi \frac{\partial A_3}{\partial z}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}\right).(A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}})$$

$$+ \phi\left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right).(A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}})$$

$$= (\nabla \phi).\mathbf{A} + \phi(\nabla.\mathbf{A})$$

Example 61

If $\mathbf{A} = \mathbf{i}y + \mathbf{j}(x^2 + y^2) + \mathbf{k}(yz + zx)$, *then find the value of curl* A *at* (1, –1, 1)

Solution

Curl $\mathbf{A} = \nabla \times \mathbf{A}$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 + y^2 & yz + zx \end{vmatrix} = \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(yz + zx) - \frac{\partial}{\partial z}(x^2 + y^2)\right] - \hat{\mathbf{j}}\left[\frac{\partial}{\partial x}(yz + zx) - \frac{\partial y}{\partial z}\right]$$

$$+ \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial y}{\partial y}\right]$$

$$= (z - 0)\hat{\mathbf{i}} - z\hat{\mathbf{j}} + (2x - 1)\hat{\mathbf{k}}$$

$$= z\hat{\mathbf{i}} - z\hat{\mathbf{j}} + (2x - 1)\hat{\mathbf{k}}$$

$$= (1)\hat{\mathbf{i}} - (1)\hat{\mathbf{j}} + [(2)(1) - 1]\hat{\mathbf{k}}$$

$$= \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Example 62

Find the value of the constant 'a' for which the vector $\mathbf{A} = \mathbf{i}(x + 3y) + \mathbf{j}(y - 2z) + \mathbf{k}(x + az)$ *is a solenoidal vector.*

Solution

A vector A is solenoidal if its divergence is zero.

$$\text{div}\,\mathbf{A} = \nabla.\mathbf{A} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).\left[\hat{\mathbf{i}}(x + y) + \hat{\mathbf{j}}(y - 2z) + \hat{\mathbf{k}}(x + az)\right]$$

$$= \frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az)$$

$= 1 + 1 + a = 0$

$a = -2$

Example 63

Evaluate div F where $\mathbf{F} = 2x^3z\hat{i} - xy^2z\hat{j} + 3y^2\hat{x}$.

Solution

$$\text{div } \mathbf{F} = \nabla . \mathbf{F} = \left(\hat{\mathrm{i}}\frac{\partial}{\partial x} + \hat{\mathrm{j}}\frac{\partial}{\partial y} + \hat{\mathrm{k}}\frac{\partial}{\partial z} \right) . (2x^3z\hat{\mathrm{i}} - xy^2z\hat{\mathrm{j}} + 3y^2x\hat{\mathrm{k}})$$

$$= \frac{\partial}{\partial x}(2x^3z) + \frac{\partial}{\partial y}(-xy^2z) + \frac{\partial}{\partial z}(3y^2x)$$

$$= 6x^2z - 2xyz + 0$$

$$= 6x^2z - 2xyz$$

Example 64

If $\mathbf{f} = xy^2\hat{\mathrm{i}} + 2x^2yz\hat{\mathrm{j}} - 3yz^2\hat{\mathrm{k}}$ *find curl* **f** *at* (1, –1, 1)

Solution

$$\text{Curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]\hat{i} - \left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]\hat{k}$$

$$= [-3z^2 - 2x^2y]\,\hat{i} - [0 + 0]\,\hat{j} + [4xyz - 2xy]\,\hat{k}$$

$$= (-3z^2 - 2x^2y)\hat{\mathrm{i}} + (0)\hat{\mathrm{j}} + (4xyz - 2xy)\hat{\mathrm{k}}$$

$$= -\hat{\mathrm{i}} - 2\hat{\mathrm{k}}$$

where we have put $x = 1$, $y = -1$ and $z = 1$.

Example 65

Show that the force $\mathbf{F} = (y^2 - x^2)\hat{\mathrm{i}} + 2xy\hat{\mathrm{j}}$ *is conservative.*

Solution

To show $\nabla \times \mathbf{F} = 0$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - x^2 & 2xy & 0 \end{vmatrix} = \hat{\mathrm{i}}\left[0 - \frac{\partial}{\partial z}(2xy)\right] - \hat{\mathrm{j}}\left[0 - \frac{\partial}{\partial z}(y^2 - x^2)\right]$$

$A_1B_1 + A_1B_1 + A_1B_1 = A_1C_1A$

$A_1B_1 = A_1C_1$

$B_1 = C_1$

$$= +\hat{k}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2 - x^2)\right]$$

$$= 0 + 0 + \hat{k}(2y - 2y)$$

$$= 0$$

Example 66

Find $\iint_S \mathbf{F}.\mathbf{n}\,ds$, *where* $\mathbf{F} = 4xzi - y^2j + yzk$ *and* S *is the surface of a cube bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0$ and $z = 1$

Solution

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla.\mathbf{F}dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right]dV$$

$$= \iiint_V (4z - y)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z - y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\Big|_{z=0}^{z=1} dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1}(2 - y)\,dy\,dx = \frac{3}{2}$$