putting $\dot{r} = \omega \times r$ yields

$$J = \int_V [r \times (\omega \times r)]\rho dV - \int_V (\dot{r} . \omega) r \rho dV.$$

1.19 GREEN'S THEOREM IN THE PLANE

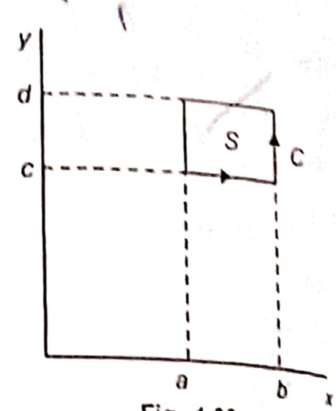


Fig. 1.30

Let $u(x, y)$ and $v(x, y)$ be functions with continuous first partial derivatives. Consider the double integral of $\frac{\partial}{\partial x} v(x, y)$ over the rectangle S as in Fig. 1.30. We shall show that the double integral is equal to a line integral around the boundary of the rectangle. We first perform the x integration to find.

$$\iint_S \frac{\partial v}{\partial x}(x, y)\, dxdy = \int_c^d \int_a^b \frac{\partial v}{\partial x}(x, y)\, dxdy = \int_c^d [v(b, y) - v(a, y)]\, dy \qquad (58)$$

We now evaluate $\oint v(x, y)\, dy$ in the counterclockwise direction so that S is always towards left as we move around the closed curve. We note that along the horizontal sides of S, integrals are zero since $dy = 0$. Along the right side, $x = b$, and y ranges from c to d. Along the left side, $x = a$, and y ranges from d to c. Hence,

$$\oint_c v(x, y) dy = \int_c^d v(b, y) dy + \int_d^c v(a, y) dy = \int_c^d [v(b, y) - v(a, y)] dy \qquad (59)$$

Combining (58) and (59),

$$\iint_S \frac{\partial v}{\partial x} dx\, dy = \oint_c v\, dy. \qquad (60)$$

Similarly,

$$-\iint_S \frac{\partial u}{\partial y} dx\, dy = \oint_c u\, dx. \qquad (60a)$$

Adding (60) and (61),

$$\oint_c (u dx + v dy) = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy \qquad (61)$$

Formula (62) is called *Green's theorem* in the plane. This result which was proved for a rectangle can be generalized to any arbitrary figure in the xy-plane bounded by a closed curve as in Fig. 1.31. Let the figure S be divided into a set of small rectangles. For each rectangle the result of (62) is valid. The double integral over the whole area S is equal to the sum of all the double integrals in (62). However, it is seen that for two typical small rectangles P and Q, the line integrals are in the opposite

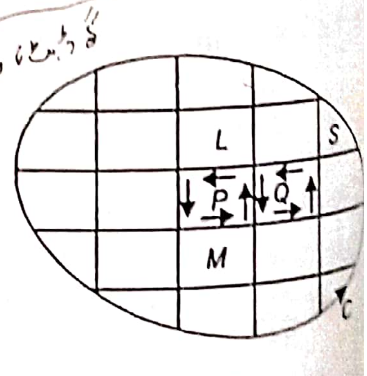
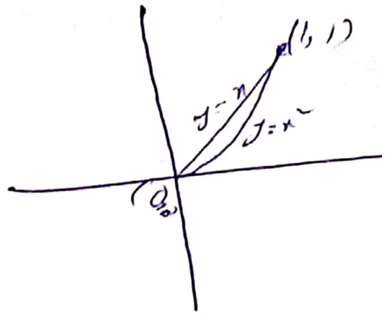


Fig. 1.31

sense over the common boundary LM and therefore get cancelled. What is left over is simply the line integral around C. Hence the result (62) is equally valid for an arbitrary figure in a plane. Thus, with the use of Green's theorem, the line integral around a closed path can be evaluated or a double integral over the area enclosed can be found out.

Example 46

Verify Green's theorem in the plane for $\oint_C (x - y)\,dx + (x + y)\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.



Let $\quad u = x - y \quad ; \quad v = x + y$

$$\frac{\partial u}{\partial y} = -1 \quad ; \quad \frac{\partial v}{\partial x} = 1$$

The curves $y = x$ and $y = x^2$ intersect at (0, 0) and (1, 1)

$$\iint\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_S [1-(-1)]\,dxdy$$

$$= 2\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} dxdy = 2\int_0^1\left[\int_{x^2}^{\sqrt{x}} dy\right]dx$$

$$= 2\int_0^1 \left(\sqrt{x} - x^2\right)dx = 2\left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$$

This is in agreement with the result obtained for the line integral in Ex. 40.

1.20 STROKES' THEOREM

S is an open two sided s. B is a simple closed curve C, Then if A has conti derivatives

This theorem states that if S is an open, two-sided surface bounded by a simple closed curve C, then, if A has continuous derivatives,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\; dS = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

where C is traversed in the positive direction.

In words, "The line integral of the tangential component of a vector A taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of A taken over any surface S having C as its boundary". This theorem relates an integral over an open surface to the line integral around the curve bounding the surface.

Proof

Let S be a surface bounded by closed contour C. Let C' be its projection on the xy-plane as in Fig. 1.32: Let a vector at each point on the surface be defined as $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$. We associate a point $P(x, y)$ on the plane with every point on the surface. Since on the surface, z is a function of x and y, that is $z = f(x, y)$, a function $A_1(x, y, z)$ becomes $A_1(x, y, z) = f(x, y)$, since the value of f must be equal to the value of A_1 on C. We may similarly consider projections on yz and xz-planes.

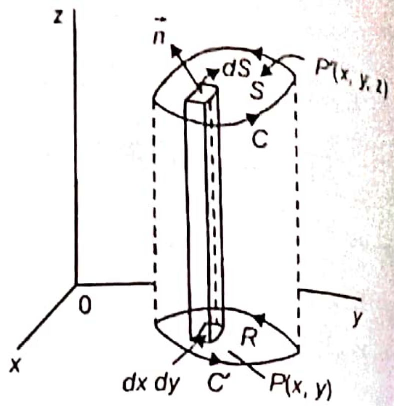


Fig. 1.32

We are required to prove that

$$\iint_S (\nabla \times \mathbf{A}).\mathbf{n}\,dS = \iint_S [\nabla \times (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}).\mathbf{n}\,dS]$$

$$= \oint_C \mathbf{A}.d\mathbf{r}$$

where C is boundary of S and $\mathbf{n}$ is a unit vector perpendicular to the surface at any point. We can find out how a typical term

$\iint_S [\nabla \times (A_1\,\mathbf{i})].\mathbf{n}\,dS$ is transformed.

$$\text{Now,}\quad \nabla \times (A_1\,\mathbf{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \mathbf{j}\frac{\partial A_1}{\partial z} - \mathbf{k}\frac{\partial A_1}{\partial y}$$

$$= [\nabla \times (A_1\mathbf{i}).\mathbf{n}\,dS] = \left(\frac{\partial A_1}{\partial z}\mathbf{n}.\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k}\right)dS \qquad (62)$$

Since $z = f(x, y)$ is taken as the equation of S, the position vector to any point of S is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$.

$$\frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

But $\dfrac{\partial \mathbf{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\mathbf{n}$, so that

$$\mathbf{n}.\frac{\partial \mathbf{r}}{\partial y} = \mathbf{n}.\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} = 0$$

or

$$\mathbf{n}.\mathbf{j} = -\frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} \qquad (63)$$

Substitute (63) in (62)

n.j

$$\left(\frac{\partial A_1}{\partial z}\mathbf{n}.\mathbf{j} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k}\right)dS = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\mathbf{n}.\mathbf{k} - \frac{\partial A_1}{\partial y}\mathbf{n}.\mathbf{k}\right)dS$$

or

$$[\nabla \times (A_1\,\mathbf{i}].\mathbf{n}dS = -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\right)\mathbf{n}.\mathbf{k}\,dS \qquad (64)$$

Now, on S,

$$A_1(x, y, z) = A_1[x, y, f(x, y)] = F(x, y)$$

Therefore,

$$\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y} \qquad (65)$$

Using (65) in (64)

$$[\nabla \times (A_1\,\mathbf{i})].\mathbf{n}\,dS = -\frac{\partial F}{\partial y}\mathbf{n}.\mathbf{k}\,dS = -\frac{\partial F}{\partial y}dx\,dy$$

Therefore, $\iint_S [\nabla \times (A_1 \mathbf{i})] \cdot \mathbf{n}\, dS = \iint_R -\frac{\partial F}{\partial y} dx\, dy$ ✓

where R is the projection of S on the xy-plane. By Green's theorem for the plane, the last integral equals $\oint_{C'} F\, dx$ where C' is the boundary of R. Since at each point (x, y) of C' the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both C and C', we must have

$$\oint_{C'} F\, dx = \oint_{C'} A_1 dx$$

or

$$\iint_S [\nabla \times (A_1 \mathbf{i})] \cdot \mathbf{n}\, dS = \oint_C A_1 dx \tag{66}$$

Similar equations are obtained by considering the projections of S on the xz and yz-planes.

$$\iint_S [\nabla \times (A_2 \mathbf{j})] \cdot \mathbf{n}\, dS = \oint_C A_2 dy \tag{67}$$

$$\iint_S [\nabla \times (A_3 \mathbf{k})] \cdot \mathbf{n}\, dS = \oint_C A_3 dz \tag{68}$$

Adding (66), (67) and (68), we get the desired result

$$\iint (\nabla \times A) \cdot \mathbf{n} dS = \oint_C A_1 dx + A_2 dy + A_3 dz = \oint_C \mathbf{A} \cdot d\mathbf{r} \tag{69}$$

Green's theorem in the plane is a special case of Stoke's theorem. ✓

1.21 DIVERGENCE THEOREM

A method of reducing triple integrals to double integrals is provided by the Divergence theorem of Gauss.

To prove: $\iiint_V \nabla \cdot A\, dV = \iint_S \mathbf{A} \cdot \mathbf{n}\, dS$

If A is the flux density of an incompressible fluid then we have shown that $\nabla \cdot A dV$ gives the total amount of fluid flowing out of the volume dV per second. The total flow from a large volume is $\iiint \nabla \cdot A dV$ which must be equal to the rate of flow across all the surfaces of the given volume $\iint \mathbf{A} \cdot d\mathbf{S}$ This is the proof of Divergence (Gauss) theorem. An analytical proof follows.

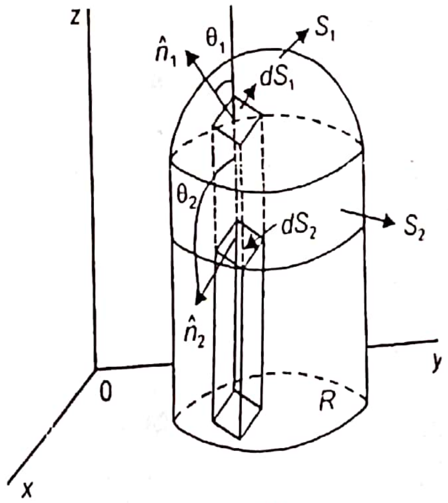


Fig. 1.33

Let S be a closed surface which is such that any line parallel to the co-ordinate axes cuts S in utmost two points. Let the equation of the upper portion S_1 be $z = f_1(x, y)$ and that of the lower portion S_2, $z = f_2(x, y)$. Let the vector field be given by $\mathbf{A} = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}$. Consider the projection of the surface on the xy-plane, called R.

$$\iiint \nabla . \mathbf{A}\, dV = \iiint \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV$$

Now,

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iiint_V \frac{\partial A_3}{\partial z} dzdxdy = \iint_R \left[\int_{f_2(x,y)}^{f_1(x,y)} \frac{\partial A_3}{\partial z} dz \right] dx\,dy$$

$$= \iint A_3(x, y, z) \Big|_{z=f_2}^{f_1} dx\,dy$$

$\iint_R \int_{f_2(x,y)}^{f_1}$

$$= \iint_R [A_3(x, y, f_1) - A_3(x, y, f_2)] dy\,dx$$

For the upper portion S_1, $dy\,dx = \cos\theta_1\, dS_1 = \mathbf{k} \,.\, \mathbf{n}_1\, dS_1$ since the normal $\mathbf{n}_1$ to S_1 makes an acute angle θ_1 with $\mathbf{k}$.

For the lower portion S_2, $dy\,dx = -\cos\theta_2\, dS_2 = -\mathbf{k} \,.\, \mathbf{n}_2\, dS_2$ since the normal $\mathbf{n}_2$ to S_2 makes an obtuse angle θ_2 with $\mathbf{k}$.

Then, $\displaystyle\iint_R A_3(x, y, f_1) dy\,dx = \iint_{S_1} A_3\, \mathbf{k} \,.\, \mathbf{n}_1\, dS_1$

$$\iint_R A_3(x, y, f_2) dy\,dx = -\iint_{S_2} A_3\, \mathbf{k} \,.\, \mathbf{n}_2\, dS_2$$

Therefore,

$$\iint_R A_3(x, y, f_1) dy\,dx - \iint_R A_3(x, y, f_2) dy\,dx$$

$$= \iint_{S_1} A_3\, \mathbf{k} \,.\, \hat{\mathbf{n}}_1\, dS_1 + \iint_{S_2} A_3\, \mathbf{k} \,.\, \hat{\mathbf{n}}_2\, dS_2$$

$$= \iint_S A_3\, \mathbf{k} \,.\, \mathbf{n}\, dS$$

So that $\displaystyle\iiint_V \frac{\partial A_3}{\partial z} dV = \iint_S A_3\, \mathbf{k} \,.\, \hat{\mathbf{n}}\, dS$ (70)

Similarly from considerations of projections of S on the yz and xz-planes we have

$$\iiint_V \frac{\partial A_1}{\partial x} dV = \iint_S A_1\, \hat{\mathbf{i}} \,.\, \hat{\mathbf{n}}\, dS \tag{71}$$

$$\iiint_V \frac{\partial A_2}{\partial y} dV = \iint_S A_2\, \mathbf{j} \,.\, \hat{\mathbf{n}}\, dS \tag{72}$$

Adding (70), (71) and (72)

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}) \,.\, \hat{\mathbf{n}}\, dS$$

or
$$\iiint_V \nabla \cdot \mathbf{A}\, dV = \iint_S \mathbf{A} \cdot \hat{\mathbf{n}}\, dS \qquad (73)$$

The Divergence theorem gives a passage from volume integral to surface integral. It is a generalization of Green's theorem in the plane and is called Green's theorem in space.

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