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Some Results on Confluent Hypergeometric Functions ${}_mF_m$

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Abstract

In this paper, we determine an integral representation of generalized hypergeometric functions ${}_{m+1}F_m$ by using Legendre's duplication formula. Moreover, we use this integral representation to obtain some results on generalized confluent hypergeometric functions ${}_mF_m$.

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1 Introduction

Following Driver and Johnston [1], the authors [2] have also recently determined the integral representation of generalized confluent hypergeometric functions as

$$ {}_mF_m\left(\frac{b}{m},\frac{b+1}{m},.....,\frac{b+m-1}{m};\frac{c}{m},\frac{c+1}{m},.....,\frac{c+m-1}{m};z\right) $$
$$ =\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt^m}dt , $$

where $\mathrm{Re}(c)>\mathrm{Re}(b)>0, m\in Z^+$.

We shall use later the Legendre's duplication formula [3] to obtain the integral representation of generalized hypergeometric functions as

$$\prod_{s=1}^{m}\Gamma\left(\alpha+\frac{s-1}{m}\right)=(2\pi)^{\frac{1}{2}(m-1)}\,m^{\frac{1}{2}-m\alpha}\,\Gamma(m\alpha). \tag{1.1}$$

2 An Integral Representation of Generalized Hypergeometric Functions

We begin with the proof of integral representation for ${}_3F_2$.

Theorem 2.1: If $\mathrm{Re}(2c-2b)>0$, then

$$ {}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};x\right) = \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-xt^2\right)^{-a}dt. \tag{2.1}$$

Proof: Suppose that $|x|<1$. Then

$$ {}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};x\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k\left(b+\frac{1}{2}\right)_k}{(c)_k\left(c+\frac{1}{2}\right)_k}\frac{x^k}{k!}$$

$$=\frac{\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+k)\Gamma\left(b+k+\frac{1}{2}\right)}{\Gamma(c+k)\Gamma\left(c+k+\frac{1}{2}\right)}\frac{x^k}{k!}.$$

$$=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(2b+2k)\Gamma(2c-2b)}{\Gamma(2c+2k)}\frac{x^k}{k!},\ \text{by (1.1)}$$

$$=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{k=0}^{\infty}(a)_k\left(\int_0^1 t^{2b+2k-1}(1-t)^{2c-2b-1}dt\right)\frac{x^k}{k!}$$

$$=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(\sum_{k=0}^{\infty}(a)_k\frac{\left(xt^2\right)^k}{k!}\right)dt$$

$$=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-xt^2\right)dt\,.$$

Theorem 2.2: If $\operatorname{Re}(2c-2b-a)>0$, then

$$ {}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right) =\frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\,{}_2F_1(a,2b;2c-a;-1)\,. \qquad (2.2)$$

Proof: Equation (2.1) yields that

$$\begin{aligned} {}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right) &=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-t^2\right)^{-a}dt\\ &=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-a-1}(1+t)^{-a}dt\\ &=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-a-1}\sum_{j_1}^{\infty}\binom{-a}{j_1}t^{j_1}dt\\ &=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{2b+j_1-1}(1-t)^{2c-2b-a-1}dt\\ &=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{\Gamma(2b+j_1)\Gamma(2c-2b-a)}{\Gamma(2c-a+j_1)}\\ &=\frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\frac{(a)_{j_1}(2b)_{j_1}}{(2c-a)_{j_1}}\frac{(-1)^{j_1}}{j_1!}\\ &=\frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\,{}_2F_1(a,2b;2c-a;-1)\,. \end{aligned}$$

Corollary 2.3: If $\operatorname{Re}(2c-2b+n)>0, n\in Z^{+}$, then

$$ {}_3F_2\left(-n,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right)=\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n,2b;2c+n;-1)\,. \qquad (2.3)$$

Theorem 2.4: If $\operatorname{Re}(3c-3b-a)>0$, then

$$ {}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right) = \frac{\Gamma(3c)\Gamma(3c-3b-a)}{\Gamma(3c-a)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(3b)_{j_1}}{(3c-a)_{j_1}} \times {}_2F_1(-j_1,3b+j_1;3c-a+j_1;-1). \quad (2.4) $$

Corollary 2.5: If $\operatorname{Re}(3c-3b+n)>0$, then

$$ {}_4F_3\left(-n,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right) = \frac{(3c-3b)_n}{(3c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}\,{}_2F_1(-j_1,3b+j_1;3c+n+j_1;-1). \quad (2.5) $$

Similar technique leads to the following generalization of 2.1

Theorem 2.6: If $\operatorname{Re}(mc-mb)>0$, then

$$ {}_{m+1}F_m\left(a,b,b+\frac{1}{m},...,b+\frac{m-1}{m};c,c+\frac{1}{m},...,c+\frac{m-1}{m};x\right) = \frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\int_0^1 t^{mb-1}(1-t)^{mc-mb-1}\left(1-xt^m\right)^{-a}dt. \quad (2.6) $$

Theorem 2.7: If $\operatorname{Re}(mc-mb-a)>0$, then

$$ {}_{m+1}F_m\left(a,b,b+\frac{1}{m},...,b+\frac{m-1}{m};c,c+\frac{1}{m},...,c+\frac{m-1}{m};1\right) = \frac{\Gamma(mc)\Gamma(mc-mb-a)}{\Gamma(mc-a)\Gamma(mc-mb)} \times\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(mb)_{j_1}}{(mc-a)_{j_1}}\prod_{r=4}^{m}\left\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{\left(mb+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}{\left(mc-a+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}\right\} \times{}_2F_1\left(-j_{m-2},mb+\sum_{k=1}^{m-2}j_k;mc-a+\sum_{k=1}^{m-2}j_k;-1\right). \quad (2.7) $$

Corollary 2.8: If $\operatorname{Re}(mc-mb+n)>0, n\in Z^+$, , then

$$
{}_{m+1}F_m\left(-n,b,b+\frac{1}{m},\dots,b+\frac{m-1}{m};c,c+\frac{1}{m},\dots,c+\frac{m-1}{m};1\right)
$$
$$
=\frac{(mc-mb)_n}{(mc)_n}
$$
$$
\times\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(mb)_{j_1}}{(mc-a)_{j_1}}\prod_{r=4}^{m}\left\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{\left(mb+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}{\left(mc+n+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}\right\}
$$
$$
\times{}_2F_1\left(-j_{m-2},mb+\sum_{k=1}^{m-2}j_k;mc+n+\sum_{k=1}^{m-2}j_k;-1\right). \qquad (2.8)
$$

3 Consequential Results on Confluent Hypergeometric Functions

Result 3.1:

$$
e^{-x}\,{}_2F_2\left(b,b+\frac{1}{2};c,c+\frac{1}{2},;x\right)
$$
$$
=\left(\sum_{n=0}^{\infty}\frac{(-x)}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(b)_k\left(b+\frac{1}{2}\right)_k}{(c)_k\left(c+\frac{1}{2}\right)_k}\frac{x^k}{k!}\right)
$$
$$
=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-x)^{n-k}}{(n-k)!}\frac{(b)_k\left(b+\frac{1}{2}\right)_k}{(c)_k\left(c+\frac{1}{2}\right)_k}\frac{x^k}{k!}
$$
$$
=\sum_{n=0}^{\infty}{}_3F_2\left(-n,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right)\frac{(-x)^n}{n!}
$$
$$
=\sum_{n=0}^{\infty}\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n,2b;2c+n;-1)\frac{(-x)^n}{n!}.
$$

It thus follows that

$$
{}_2F_2\left(b,b+\frac{1}{2};c,c+\frac{1}{2},;x\right)=e^{x}\sum_{n=0}^{\infty}\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n,2b;2c+n;-1)\frac{(-x)^n}{n!}.
$$

Result 3.2:

$$e^{-x}\,{}_3F_3\left(b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(b)_k\left(b+\frac{1}{3}\right)_k\left(b+\frac{2}{3}\right)_k}{(c)_k\left(c+\frac{1}{3}\right)_k\left(c+\frac{2}{3}\right)_k}\frac{x^k}{k!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{(-n)_k(b)_k\left(b+\frac{1}{3}\right)_k\left(b+\frac{2}{3}\right)_k}{(c)_k\left(c+\frac{1}{3}\right)_k\left(c+\frac{2}{3}\right)_k}\frac{1}{k!}\right)\frac{(-x)^n}{n!}$$

$$=\sum_{n=0}^{\infty}{}_4F_3\left(-n,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right)\frac{(-x)^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(3c-3b)_n}{(3c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}$$

$$\times\,{}_2F_1(-j_1,3b+j_1;3c+n+j_1;-1)\frac{(-x)^n}{n!},\text{ by 2.6.}$$

Hence, it follows that

$${}_3F_3\left(b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)$$

$$=e^x\sum_{n=0}^{\infty}\frac{(3c-3b)_n}{(3c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}$$

$$\times\,{}_2F_1(-j_1,3b+j_1;3c+n+j_1;-1)\frac{(-x)^n}{n!}.$$

We Generalize 3.1 and 3.2 as

Result 3.3: For $m>3$, we have

$$e^{-x}\,{}_mF_m\left(b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right)$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-x)}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(b)_k\left(b+\frac{1}{m}\right)_k\ldots\left(b+\frac{m-1}{m}\right)_k}{(c)_k\left(c+\frac{1}{m}\right)_k\ldots\left(c+\frac{m-1}{m}\right)_k}\frac{x^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-x)^{n-k}}{(n-k)!}\frac{(b)_k\left(b+\frac{1}{m}\right)_k\cdots\left(b+\frac{m-1}{m}\right)_k}{(c)_k\left(c+\frac{1}{m}\right)_k\cdots\left(c+\frac{m-1}{m}\right)_k}\frac{x^k}{k!}$$

$$= \sum_{n=0}^{\infty}{}_{m+1}F_m\left(-n,b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};1\right)\frac{(-x)^n}{n!}.$$

So that

$${}_mF_m\left(b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right)$$

$$= e^x\sum_{n=0}^{\infty}\frac{(mc-mb)_n}{(mc)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(mb)_{j_1}}{(mc-a)_{j_1}}$$

$$\times\prod_{r=4}^{m}\left\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{\left(mb+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}{\left(mc+n+\sum_{s=1}^{r-3}j_s\right)_{j_{r-2}}}\right\}$$

$$\times{}_2F_1\left(-j_{m-2},mb+\sum_{k=1}^{m-2}j_k;mc+n+\sum_{k=1}^{m-2}j_k;-1\right)\frac{(-x)^n}{n!}.$$

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