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$k-$ANALOGUE OF KUMMER'S FIRST FORMULA

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Abstract. The aim of this research article is to obtain an $k-$analogue of Kummer's first formula ${}_1F_1\left[\begin{array}{c} a \\ \quad ;x \\ b \end{array}\right] = e^x\ {}_1F_1\left[\begin{array}{c} a \\ \quad ;-x \\ b-a \end{array}\right]$ and evaluate the very useful formula ${}_2F_{1,k}\left[\begin{array}{c} (-n,1),(a,k) \\ \quad ;1 \\ (b,k) \end{array}\right] = \frac{(b-a)_{n,k}}{(b)_{n,k}}$ by using the integral representation for $k-$hypergeometric functions.

1. Introduction

Diaz et al. [1, 2, 3] have introduced $k-$gamma and $k-$beta functions and proved a number of their properties. They have also studied $k-$zeta functions and $k-$hypergeometric functions based on Pochhammer $k-$symbols for factorial functions. These studies were then followed by works of Mansour [7], Kokologiannaki [4], Krasniqi [5,6] and Merovci [8] elaborating and strengthening the scope of $k-$gamma and $k-$beta functions. Very recently, Mubeen and Habibullah [9] defined $k-$fractional integration and gave an its application. Mubeen and Habibullah [10] also introduced an integral representation of some generalized confluent $k-$hypergeometric functions ${}_mF_{m,k}$ and $k-$hypergeometric functions ${}_{m+1}F_{m,k}$ by using the properties of Pochhammer $k-$symbols, $k-$gamma and $k-$beta functions as

$$
{}_mF_{m,k}\left[\begin{array}{c} (\frac{\beta}{m},k),(\frac{\beta+1}{m},k),...,(\frac{\beta+m-1}{m},k) \\ \quad ;x \\ (\frac{\gamma}{m},k),(\frac{\gamma+1}{m},k),...,(\frac{\gamma+m-1}{m},k) \end{array}\right]
$$

$$
= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}dt \qquad (1.1)
$$

and

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$$
{}_{m+1}F_{m,k}\left[\begin{array}{c} (\alpha,1)(\frac{\beta}{m},k),(\frac{\beta+1}{m},k),...,(\frac{\beta+m-1}{m},k) \\ \\ (\frac{\gamma}{m},k),(\frac{\gamma+1}{m},k),...,(\frac{\gamma+m-1}{m},k) \end{array} ;x\right]
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-xt^m)^{-\alpha}dt. \quad (1.2)
$$

2. Main Results

In this section, we find out the main results.

Theorem 2.1. *If* $Re(\gamma-\beta)>0, k>0$*, then*

$$
{}_2F_{1,k}\left[\begin{array}{c} (\alpha,1),(\beta,k) \\ \\ (\gamma,k) \end{array} ;1\right]=\frac{\Gamma_k(\gamma)\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-\beta)\Gamma_k(\gamma-k\alpha)}. \quad (2.1)
$$

Proof. From equation (1.2), we have the following result

$$
{}_2F_{1,k}\left[\begin{array}{c} (\alpha,1),(\beta,k) \\ \\ (\gamma,k) \end{array} ;x\right]
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-xt)^{-\alpha}dt. \quad (2.2)
$$

Substitute $x=1$ in equation (2.2), we obtain the following form

$$
{}_2F_{1,k}\left[\begin{array}{c} (\alpha,1),(\beta,k) \\ \\ (\gamma,k) \end{array} ;1\right]
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-\alpha-1}dt.
$$

Hence, we find the required result by using the definition of $k-$beta functions.

□

Proposition 2.2. *If* $\gamma=b,\beta=a,\alpha=-n,k>0$*, then*

$$
{}_2F_{1,k}\left[\begin{array}{c} (-n,1),(a,k) \\ \\ (b,k) \end{array} ;1\right]=\frac{(b-a)_{n,k}}{(b)_{n,k}}. \quad (2.3)
$$

Theorem 2.3. *If* $Re(\gamma-\beta)>0, k>0$*, then*

$$
{}_1F_{1,k}\left[\begin{array}{c} (a,k) \\ \\ (b,k) \end{array} ;x\right]=e^x\,{}_1F_{1,k}\left[\begin{array}{c} (a,k) \\ \\ (b-a,k) \end{array} ;-x\right]. \quad (2.4)
$$

Proof. Consider the product of exponential function e^{-x} and confluent $k-$hypergeometric functions ${}_1F_{1,k}$ as

$$e^{-x}\ {}_1F_{1,k}\left[\begin{array}{cc}(a,k) & \\ & ;x \\ (b,k) & \end{array}\right]$$

$$= [\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}][\sum_{n=0}^{\infty}\frac{(a)_{n,k}x^n}{(b)_{n,k}n!}]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n-m}(a)_{m,k}x^n}{(b)_{m,k}m!(n-m)!}$$

$(see\ \ Rainville\ \ [11])$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n(-n)_m(a)_{m,k}x^n}{(b)_{m,k}m!n!}$$

$$= \sum_{n=0}^{\infty}[\sum_{m=0}^{n}\frac{(-n)_{m,1}(a)_{m,k}}{(b)_{m,k}m!}]\frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\ {}_2F_{1,k}\left[\begin{array}{cc}(-n,1),(a,k) & \\ & ;1 \\ (b,k) & \end{array}\right]\frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(b-a)_{n,k}(-x)^n}{(b)_{n,k}n!}.$$

Hence, we obtain the equation (2.4) by using the equation (2.3)and the definition of $k-$hypergeometric functions.

□

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