Mechanics → "deals with machines"

⇒ Mechanics is the branch of Science. Which deals with the motion of objects.

Types:-

i) Classical Mechanics (deals with ordinary obj.

ii) Quantum Mechanics (" Microscopic objects

Classical Mechanics:- → (mainly used)

⇒ It deals with the objects which are usually visible by naked eye.

Quantum Mechanics:-

⇒ It deals with microscopic objects.

Particle:-

A particle is a point mass i.e a definite mass with no size (no dimension i.e length height etc) & it is an imaginary (abstract) Concept.

Rigid body:-

A rigid body is a Collection of Such particles, distance b/w them is unchanged even under the effect of any type of external force &

A rigid body obeys the Constraint

$$|r_i - r_j|^2 = c_{ij} \quad \text{(Constant)}$$

Where r_i is a position vector of ith particle of rigid body.

Newtonian Mechanics:-

⇒ It is that type of mechanics where Newton's law of motion remain valid. (i.e it deals with Newton's laws).

Generalized form:-

It is more generalized in the form of Hamiltonian & Lagrangian Mechanics.

Newton's law of Gravitation:-

$$F = G\frac{m_1 m_2}{r^2 = d^2}$$

$d^2 \rightarrow$ distance b/w their centres

Inertial frame of Reference:-

$a = 0$

⇒ It is frame of reference in which Newton's law of motion hold. These frame of reference are either at rest or moving with uniform velocity.

⇒ Earth is considered as inertial frame irrespective of that it is rotating.

⇒ Practically it is due to small distance b/w the objects as compared to rotation of earth & small velocities of objects as compared to the velocity of earth.

⇒ History of Mechanics
Aristotle's Theory
Mechanics during middle ages
Keplar Galilio & Newton's Period.
Hamilton, Lagrange.

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Inertia:-

Inertia is the resistance of any physical object to any change in its state of motion including changes to its speed & direction.
It is tendency of object to keep moving in a straight line at constant velocity i.e first law of Newton is inertia.

Moment of Inertia:-

$$I = \frac{L}{\omega}$$

$L \rightarrow$ angular momentum
$\omega \rightarrow$ angular velocity.

Moment of inertia is mass property of rigid body that determines the torque needed for a desired angular acceleration about axis of rotation.

Angular Momentum:-

⇒ It is measure of the amount of rotation of object & has product

$$L = I\omega$$

$L \rightarrow$ linear momentum

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Translation & rotation:-

The displacement of a rigid body is a change in position of its particles in any direction. This displacement is called translation. If all particles of the body are displaced by equal amount then vectors joining initial & final positions of the particles are mutually parallel.
If all the particles of the body displaces through some angle around a fixed line (axis), displacement is called rotation.
In rotation, motion of each particle traces an arc of circle.

Angular Velocity:-

The rate of change of angular displacement of a rigid body.
If $\Delta\theta$ is angular displacement of rigid body in a very small interval of time Δt, then angular speed is defined as.

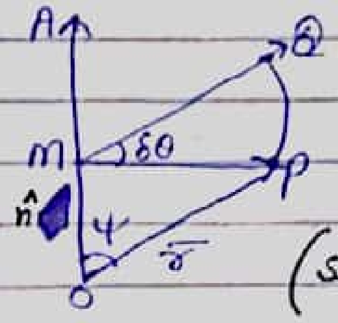
$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta}$$

If $\hat{n}$ is a unit vector in the direction of axis of rotation. Then the angular velocity is

$$\bar{\omega} = \dot{\theta}\hat{n}$$

let a particle rotates about a fixed axis OA. & $\bar{r}$ is position-vector of the particle P making an angle ψ with the axis OA.

Let the particle rotates about the axis through an angle $\delta\theta$ & displaces through small arc $\widehat{PQ}$ then in limiting case;



(s=vt

$\widehat{PQ} = v\delta t \rightarrow (1)$

also, $\widehat{PQ} = |PM|\delta\theta$ (∵ $l = r\theta$

$= r\sin\psi\,\delta\theta \rightarrow (2)$

by (1) & (2)

$v\delta t = r\sin\psi\,\delta\theta \rightarrow (3)$

$|\hat{n}\times\bar{r}| = |\hat{n}||\bar{r}|\sin\psi$

$|\hat{n}\times\bar{r}| = r\sin\psi$

(3) becms,

$v\delta t = |\hat{n}\times\bar{r}|\delta\theta$

$v = |\hat{n}\times\bar{r}|\frac{\delta\theta}{\delta t}$

$= |\hat{n}\times\bar{r}| \lim_{\delta t\to 0}\frac{\delta\theta}{\delta t}$

$= |\hat{n}\times\bar{r}|\frac{d\theta}{dt} = \dot{\theta}|\hat{n}\times\bar{r}|$

$= |\dot{\theta}\hat{n}\times\bar{r}|$

In vector form,

$\bar{v} = \bar{\omega}\times\bar{r}$

<u>Angular accelration:-</u>

Rate of Change of angular velocity.

$\bar{a} = \frac{d\bar{\omega}}{dt} = \frac{d^2\theta}{dt^2}$

Angular Momentum of a particle

⟹ Moment of linear momentum is called angular momentum.

⟹ If m is mass of the particle & $\bar{v}$ is its velocity then its momentum is given by $\underline{p} = m\underline{v}$

So, moment of this momentum (linear) is given by $\underline{L} = \underline{r} \times \underline{p}$
Where $\underline{r}$ is the position vector (radius vector) of the particle & $\underline{L}$ is angular momentum.

We have,

$$\underline{v} = \underline{\omega} \times \underline{r}$$

So,

$$\underline{L} = \underline{r} \times \underline{p}$$

$$= \bar{r} \times m\bar{v} = m\,\bar{r} \times \bar{v}$$

$$= m\,\bar{r} \times (\bar{\omega} \times \bar{r})$$

$$= m[(\bar{r}\cdot\bar{r})\bar{\omega} - (\bar{r}\cdot\bar{\omega})\underline{r}]$$

$$\underline{L} = m[r^2\bar{\omega} - \bar{r}\cdot\bar{\omega}\,\bar{r}]$$

$(\because \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c})$

τ = moment $= \bar{r} \times \bar{F}$

The quantity angular momentum can also be represented by 'J'.

Angular Momentum of a System of Particles-

Consider a system of n particles having masses $m_1, m_2, \dots m_n$. S.t. their respective radius vectors are given by $r_1, r_2, r_3 \dots r_n$. The eq/n of motion for any ith particle is

$$\bar{F_i} = m a_i = m \frac{d^2 \bar{r_i}}{dt^2}$$

& $\quad \bar{v_i} = \bar{\omega} \times \bar{r_i}$

Where each particle is assumed to have same angular velocity.

angular momentum $= \underline{L_i} = \bar{r_i} \times m_i \bar{v_i}$

$$= m_i \bar{r_i} \times \bar{v_i}$$

$$= m_i \bar{r_i} \times (\bar{\omega} \times \bar{r_i})$$

Total angular momentum for the System of particles.

$$\underline{L} = \sum_{i=1}^{n} L_i$$

$$= \sum_{i=1}^{n} \left[m_i \bar{r_i} \times (\bar{\omega} \times \bar{r_i}) \right]$$

$$= \sum_{i=1}^{n} m_i \left[r_i^2 \bar{\omega} - \bar{r_i} . \bar{\omega r_i} \right] \rightarrow ①$$

Where,

$$\bar{r_i} = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$

$$\underline{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\bar{\omega} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

① becms,

$$L_x\hat{i}+L_y\hat{j}+L_z\hat{k}=\sum m_i[(x_i^2+y_i^2+z_i^2)(\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k})$$
$$-(x_i\omega_x+y_i\omega_y+z_i\omega_z)(x_i\hat{i}+y_i\hat{j}+z_i\hat{k})]$$

Comparing coefficients of $\hat{i}$

$$L_x=\sum m_i[(x_i^2+y_i^2+z_i^2)\omega_x-(x_i\omega_x+y_i\omega_y+z_i\omega_z)x_i]$$
$$=\sum m_i[\cancel{x_i^2\omega_x}+(y_i^2+z_i^2)\omega_x-\cancel{x_i^2\omega_x}-x_iy_i\omega_y-x_iz_i\omega_z]$$

$$=\sum m_i[(y_i^2+z_i^2)\omega_x-x_iy_i\omega_y-x_iz_i\omega_z]$$
$$=\sum m_i(r_i^2-x_i^2)\omega_x-\sum m_ix_iy_i\omega_y-\sum m_ix_iz_i\omega_z$$

$$L_x=I_{xx}\omega_x+I_{xy}\omega_y+I_{xz}\omega_z$$

$\because r_i^2=x_i^2+y_i^2+z_i^2$

$r_i^2-x_i^2=y_i^2+z_i^2$

Similarly;

$$L_y=I_{yx}\omega_x+I_{yy}\omega_y+I_{yz}\omega_z$$

&

$$L_z=I_{zx}\omega_x+I_{zy}\omega_y+I_{zz}\omega_z$$

Where,

$$I_{xx}=\sum m_i(r_i^2-x_i^2)$$

$$I_{yy}=\sum_{m_i} m_i(r_i^2-y_i^2)$$

&

$$I_{zz}=\sum m_i(r_i^2-z_i^2)$$

$$I_{xy}=-\sum m_ix_iy_i=I_{yx}$$

$$I_{xz}=-\sum m_ix_iz_i=I_{zx}$$

&

$$I_{yz}=-\sum m_iy_iz_i=I_{zy}$$

$I_{xx} = I_{11}$
$I_{yy} = I_{22}$
$I_{zz} = I_{33}$

These are moment of inertia & product of inertia respectively.
i.e

$I_{xx}, I_{yy}, I_{zz} \longrightarrow$ moment of Inertia,

$I_{xy}, I_{yz}, I_{zx} \longrightarrow$ product of Inertia.

Moment of Inertia for a rigid body

We have calculated moment of Inertia for a system of n-particles i.e

$$I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2) \longrightarrow (1)$$

A rigid body can be considered as a collection of infinite many infinitesimal (small) particles. Particles with no gap b/w them. so, considering the mass of the infinitesimal ith particle $\Delta m_i = \rho(r_i)\Delta V_i$; where ΔV_i is the volume of the ith particle & $\rho(r_i)$ is the density of the particle.

from (1), we have,

$$I_{xx} = \sum_{i=1}^{n} \Delta m_i (y_i^2 + z_i^2)$$

$$= \sum_{i=1}^{n} (y_i^2 + z_i^2) \rho(r_i) \Delta V_i$$

for whole rigid body taking $\lim_{n \to \infty}$, we get,

$$I_{xx} = \lim_{n \to \infty} \sum_{i=1}^{n} (y_i^2 + z_i^2) \rho(r_i) \Delta V_i$$

Similarly, $I_{xx} = \int_V (y_i^2 + z_i^2)\, \rho(r)\, dv$

$$I_{yy} = \int_V (z^2 + x^2)\, \rho(r)\, dv$$

$$I_{zz} = \int_V (x^2 + y^2)\, \rho(r)\, dv$$

also, Corresponding Product of inertia are, $I = -\sum x_i y_i m_i = -\int xy\rho\, dv$

$$\vec{I_{xy}} = -\int_V \rho(r)\, xy\, dv$$

$$I_{yz} = -\int \rho(r)\, yz\, dv = I_{zy}$$

$$I_{zx} = -\int xz\, \rho(r)\, dv = I_{xz}$$

Note

in discrete system $\xrightarrow{\text{notation}}$ $\sum m_i d_i$

" Continous system $\longrightarrow \int$

$$I_{xx} = \sum m_i (y_i^2 + z_i^2) = \sum_{i=1}^{n} m_i (r_i^2 - x_i^2) m_i = \sum_{i=1}^{n} m_i d_i$$

" Moment of Inertia of a rigid body about a given lines "

let M be the mass of the system & $\hat{e}$ be the unit vector along the line, along which we have to