find moment of inertia.

Consider,

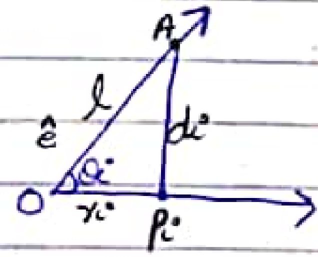
$$\hat{e} = \lambda\hat{i} + \mu\hat{j} + \gamma\hat{k} \quad \xrightarrow{} \text{new}$$

Where $(\lambda, \cancel{\mu}, \mu, \gamma)$ are direction cosine of the line.

If m_i is the mass of ith particle of the system & d_i is ⊥ distance from P_i to the line l, then,

$$I_l = \sum_{i=1}^{n} m_i d_i^2 \longrightarrow (1)$$

$$\sin\theta_i = \frac{d_i}{|OP_i|} = \frac{\perp}{h} = \frac{d_i}{r_i}$$



$$\Rightarrow d_i = r_i \sin\theta_i$$

(∵ base = line, ⊥ = ⊥ distr, h = OP_i)

also,

$$|\hat{e} \times \bar{r}_i| = |\hat{e}||\bar{r}_i|\sin\theta_i = r_i \sin\theta_i$$

So, $d_i = |\hat{e} \times \bar{r}_i|$

hence,

$$I_l = \sum_{i=1}^{n} m_i |\hat{e} \times \bar{r}_i|^2$$

Now

(∵ $\hat{e} = \lambda\hat{i} + \mu\hat{j} + \gamma\hat{k}$, $\bar{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$)

$$\hat{e} \times \bar{r}_i = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda & \mu & \gamma \\ x_i & y_i & z_i \end{vmatrix}$$

$$\hat{e} \times \bar{r}_i = \hat{i}[\mu z_i - \gamma y_i] - \hat{j}[\lambda z_i - \gamma x_i] + \hat{k}[\lambda y_i - \mu x_i]$$

$$|\hat{e} \times \bar{r}_i|^2 = (\mu z_i - \gamma y_i)^2 + (\lambda z_i - \gamma x_i)^2 + (\lambda y_i - \mu x_i)^2$$

$$I_l = \sum_{i=1}^{n} m_i[(\mu z_i - \gamma y_i)^2 + (\lambda z_i - \gamma x_i)^2 + (\lambda y_i - \mu x_i)^2]$$

$$= \sum m_i[\mu^2 z_i^2 + \gamma^2 y_i^2 - 2\mu z_i \gamma y_i + \lambda^2 z_i^2 + \gamma^2 x_i^2 - 2\lambda\gamma z_i x_i$$

$$+\lambda^2 y_i^2 + \mu^2 x_i^2 - 2\lambda y_i \mu x_i]$$

$$= \sum m_i [\mu^2(x_i^2+z_i^2) + \lambda^2(y_i^2+z_i^2) + \nu^2(y_i^2+x_i^2) - 2\mu\nu z_i y_i - 2\lambda\nu z_i x_i - 2\lambda\mu y_i x_i]$$

$$= \mu^2 \sum m_i (x_i^2+z_i^2) + \lambda^2 \sum m_i (y_i^2+z_i^2) + \nu^2 \sum m_i (y_i^2+x_i^2) + 2\mu\nu(-\sum m_i z_i y_i) + 2\lambda\nu(-\sum m_i x_i z_i) + 2\mu\lambda(-\sum m_i x_i y_i)$$

$$= \mu^2 I_{yy} + \lambda^2 I_{xx} + \nu^2 I_{zz} + 2\lambda\nu I_{zx} + 2\mu\lambda I_{xy} + 2\nu\mu I_{yz}$$

$$I_l = \lambda^2 \overset{I_{11}}{I_{xx}} + \mu^2 \overset{I_{22}}{I_{yy}} + \nu^2 \overset{I_{33}}{I_{zz}} + 2\lambda\mu \overset{I_{12}}{I_{xy}} + 2\mu\nu \overset{I_{23}}{I_{yz}} + 2\nu\lambda \overset{I_{31}}{I_{zx}}$$

is moment of Inertia of a rigid body about a line.

Note 2nd, d-cosines of λ, μ, ν Coordi Σe

$\lambda = \cos\alpha = 90 = 0$

$\mu = \cos\beta = 90 = 0$

$\nu = \cos\gamma = 0 = 1$

Component itself is direction ratio

$a = a_1 \hat{i}$

Parallel Axis Theorem:

Statement:-

The moment of inertia of a rigid body about a given line is equal to the ~~as~~ sum of moment of inertia about a ∥ axis through centroid & moment of inertia of the total mass placed at the centroid.

Proof:-

If I denotes the moment of inertia about a given axis OA (say) & d_i represents ⊥ distance of the ith particle from the axis then,

$$I = \sum_{i=1}^{n} m_i d_i^2$$

$$= \sum_{i=1}^{n} m_i (\hat{e} \times \underline{r_i})^2 \rightarrow ①$$

Where $\hat{e}$ is the unit vector along the axis OA.

Let C be the centroid of the rigid body & r_c be its p.v. Draw an axis CA′ ∥ to OA through C. Let r_i' be the p.v of P_i w.r.t centroid C. then,

$$\underline{r_i} = \underline{r_c} + \underline{r_i'}$$

$$I = \sum_{i=1}^{n} m_i [\hat{e} \times (\underline{r_c} + \underline{r_i'})]^2$$

$$= \sum_{i=1}^{n} m_i [\hat{e} \times \underline{r_c} + \hat{e} \times \underline{r_i'}]^2$$

$$= \sum_{i=1}^{n} m_i [(\hat{e} \times \underline{r_c})^2 + (\hat{e} \times \underline{r_i'})^2 + 2(\hat{e} \times \underline{r_c}) \cdot (\hat{e} \times \underline{r_i'})]$$

$$= \sum_{i=1}^{n} m_i (\hat{e} \times \underline{r_c})^2 + \sum_{i=1}^{n} m_i (\hat{e} \times \underline{r_i'})^2 + 2\sum_{i=1}^{n} m_i (\hat{e} \times \underline{r_c}) \cdot (\hat{e} \times \underline{r_i'})$$

$$= \sum_{i=1}^{n} m_i d_c^2 + \sum_{i=1}^{n} m_i d_i'^2 + 2(\hat{e} \times \underline{r_c}) \cdot e \times \sum_{i=1}^{n} m_i \underline{r_i'}$$

$$= M d_c^2 + \sum_{i=1}^{n} m_i d_i'^2 + 0$$

$$= M d_c^2 + \sum_{i=1}^{n} m_i d_i'^2$$

$$I = I_0 + I'$$

* C.O.M $= \underline{r} = \dfrac{\sum m_i r_i}{\sum m_i}$ ↓ p.v of C.O.M

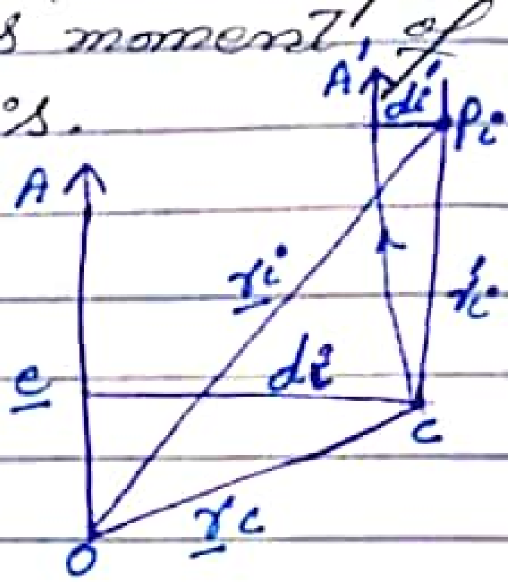
* if com is at origin then $r = 0$

$\sum m_i r_i = 0$

$m_1 + m_2 \ldots = M$

but $m_1 d_1 + \ldots + M$

Where I_0, is moment of inertia of Centroid & $I' = \sum_{i=1}^{n} m_i d_i'^2$ is moment of inertia of P_i w.r.t $\parallel$ axis.



Cartesian Coordinates:-

$$I_{11} = I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$$
$$= \sum_{i=1}^{n} m_i (r_i^2 - x_i^2)$$

is m.g of P_i about x-axis w.r.t origin 'O'.

So, $\sum_{i=1}^{n} m_i (r_c^2 - x_c^2)$ or

$M(y_c^2 + z_c^2)$ is m.g of centroid C about x-axes w.r.t 'O'. & taking as origin

I_{c11} is m.g of P_i w.r.t centroid $\vec{C}$ about an axis $\parallel$ to x-axis

hence, $I = M(y_c^2 + z_c^2) + I_{c11}$

Similarly,

$$I_{22} = M(z_c^2 + x_c^2) + I_{c22} \rightarrow M(r_c^2 - y_c^2)$$
$$I_{33} = M(x_c^2 + y_c^2) + I_{c33}$$
$$I_{12} = -M x_c y_c + I_{c12} = I_{21}$$
$$I_{23} = -M y_c z_c + I_{c23} = I_{32}$$
$$I_{31} = -M z_c x_c + I_{c31} = I_{13}.$$

Perpendicular Axis Theorem

Statement:-

The moment of inertia of a plane rigid body about an axis ⊥ to the rigid body is equal to the sum of moments of inertia about two mutually ⊥ axis lying in the plane of the rigid body & intersecting at the common point with the given axis.

Proof:-

If we choose plane rigid body in xy plane & z-axis ⊥ to the rigid body, then the above statement implies

$$I_{zz} = I_{xx} + I_{yy} \quad \text{or} \quad I_{33} = I_{11} + I_{22}$$

Where $I_{11} = I_{xx}$, $I_{22} = I_{yy}$ & $I_{33} = I_{zz}$ are moments of inertia about x-axis, y-axis, z-axis.

By choosing z-axis as a ⊥ axis to the lamina (plane) rigid body, we have,

$$I_{33} = I_{zz} = \sum_{i=1}^{n} m_i d_i^2 \longrightarrow (i)$$

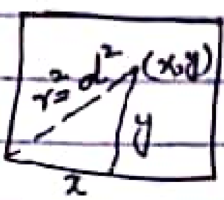
Where d_i is the ⊥ distance of the ith particle from the base of z-axis.

using $d_i^2 = x_i^2 + y_i^2$

x is ⊥ distance from y axis

$$\text{So,} \quad I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$$

$$= \sum_i m_i x_i^2 + \sum m_i y_i^2$$



$d_i^2 = x_i^2 - z_i^2$
$= x_i^2 - 0$
$= x_i^2$
$= x_i^2 + y^2$

z-axis is zero in x-y plane

Similarly $= I_{xx} + I_{yy}$

$$I_{xx} = I_{yy} + I_{zz}$$

$$I_{yy} = I_{xx} + I_{zz}$$

Proved

P-16

Rotational K.E of a rigid body

Suppose, we have a rigid body of a large no. of particles, let $m_1, m_2, \ldots, m_n$ be the masses & $r_1, r_2, \ldots, r_n$ be the position vectors of these particles respectively.
Let v_i be the velocity of the ith particle then, $\underline{v} = \underline{\omega} \times \underline{r_i}$
Where, $\underline{\omega}$ is angular velocity of the rigid body.
$\therefore$ The energy due to motion of the body (K.E) is equal to the sum i.e. of K.E's of all particles constituting the rigid body.

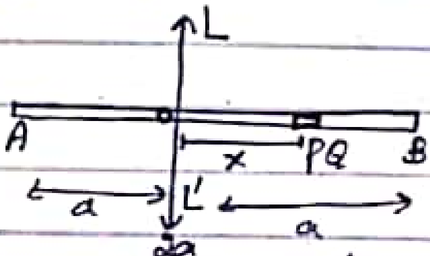
i.e
$$K.E = \tfrac{1}{2} \sum m_i v_i^2$$
$$= \tfrac{1}{2} \sum m_i |\underline{\omega} \times \underline{r_i}|^2$$
$$= \tfrac{1}{2} \sum m_i (\underline{\omega} \times \underline{r_i}) \cdot (\underline{\omega} \times \underline{r_i})$$
$$= \tfrac{1}{2} \sum m_i (\underline{\omega} \times \underline{r_i}) \times (\omega \times r_i)$$
$$= \tfrac{1}{2} \sum m_i \, \underline{\omega} \cdot \underline{r_i} \times \underline{v_i}$$
$$= \tfrac{1}{2} \, \underline{\omega} \cdot \sum m_i \underline{r_i} \times \underline{v_i}$$
$$= \tfrac{1}{2} \underline{\omega} \cdot \sum \underline{r_i} \times \underline{p_i}$$
$$= \tfrac{1}{2} \underline{\omega} \cdot \sum_i \underline{L_i}$$
$$= \tfrac{1}{2} \underline{\omega} \cdot \underline{L}$$ is Rotational K.E when $L = \sum L_i$ is angular mom. of the rigid body.

M.I of a uniform rod.

Sol:

let $AB = 2a$ be the length of the uniform rod of density ρ. we find its M.I about a $\perp$ axes LOL' passing through its mid pt. we take an element PQ of the rod at a distance from O then mass of element $= \Delta m = \rho \Delta x$

$$\Delta m = \frac{M}{2a} \Delta x.$$



$\therefore$ M.I about LOL'

$$\therefore I_{yy} = \lim_{n \to \infty} \sum_{i=1}^{n} m_i d_i^2$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta m_i x_i^2$$

$$= \int_{-a}^{a} x^2 \, dm$$

$$= \int_{-a}^{a} x^2 \frac{M}{2a} dx = \frac{M}{2a} \left| \frac{x^3}{3} \right|_{-a}^{a}$$

$$= \frac{M}{2a} \frac{2a^3}{3} = \frac{1}{3} Ma^2$$

" $\rho \to$ density
$\rho = M/V$
$\Rightarrow M = \rho V$.
$dm = \rho dv$
in case of length of rod
$dm = \rho dx$.

* When we take M.I at the m.pt of rod then it is $\frac{1}{3} Ma^2$.

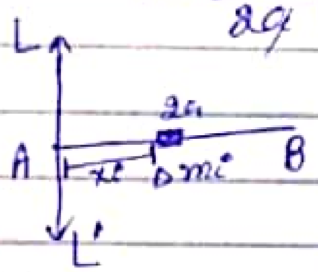
ii) M.I of the same rod at one end of the rod.

We have to find M.I at one end say A, about a $\perp$ axes LAL'. Taking A as origin & LAL' as y-axes. if Δm_i is mass of the ith particle & x_i is its distance from origin A. then

$$M.I = \int_0^{2a} x^2\,dm = \int_0^{2a} x^2 \rho\,dx$$

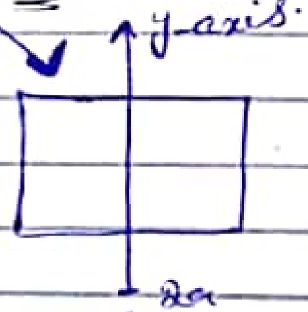
$$= \int_0^{2a} x^2 \frac{M}{2a}\,dx = \frac{M}{2a}\left|\frac{x^3}{3}\right|_0^{2a}$$

$$= \frac{M}{2a}\left(\frac{8a^3}{3}\right) = \frac{4Ma^2}{3} = \frac{4}{3}Ma^2$$

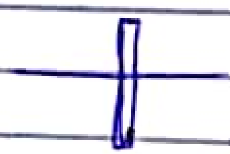


*— uniform rectangular lamina

$$I_{zz} = I_{xx} + I_{yy}$$
$$= \tfrac{1}{3}Mb^2 + \tfrac{1}{3}Ma^2$$
$$= \tfrac{1}{3}M(a^2+b^2)$$



$I_{yy} = \frac{1}{3}Ma^2$



$I_{xx} = \frac{1}{3}Mb^2$

ex/2:- find M.I of a uniform ring about an axis through its centre.

let M be the mass & a be the radius of the ring & circumference of the ring $= 2\pi a$. If δs is arc length of an element of the ring,

$[I = \sum m_i d_i^2]$

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} a^2 \rho\,\delta s$$

$$= \int_0^{2\pi a} a^2 \rho\,ds = a^2\rho \int_0^{2\pi a} ds.$$

In 3D → volume. $\rho = \frac{M}{V}$

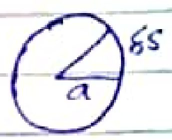
In 2D → Area

In 1D → length

$\rho = \frac{M}{2\pi a}$

$= a^2 \rho |S|_0^{2\pi a} = 2a^3\pi\rho$

$= 2a^3\pi \cdot \frac{M}{2\pi a} = Ma^2$



$c = 2\pi r$

$\delta - v = l = \delta s$ & $a = di$ ⊥ distnc or radius

exp 3:

M.I of a circular disc of given mass & radius about an axis through its centre & ⊥ to plane.

Sol.

we have M & 'a' as the mass & radius of a circular disc with density $\rho = \frac{M}{A} = \frac{M}{\pi a^2}$

we assume, the disc is composed of concentric circular strips of radii 'r_i' & thickness/width δr_i, then mass of the ith strip $= \rho 2\pi r_i \delta r_i = \delta m_i$

∴ M.I of the ith strip $= r_i^2 \delta m_i$

" " of disc $= \lim_{n\to\infty} \sum_{i=1}^{n} r_i^2 \delta m_i$

$= \lim_{n\to\infty} \sum_{i=1}^{n} r_i^2 \frac{M}{\pi a^2}(2\pi r_i \delta r_i)$

$= 2\pi \frac{M}{\pi a^2} \int_0^a r^3 dr$

$= \frac{2M}{a^2}\left[\frac{r^4}{4}\right]_0^a = \frac{M}{2a^2}(a^4)$

$= \frac{Ma^2}{2}$

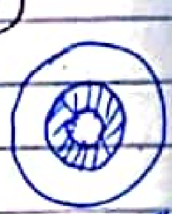
ρV → area here.

$M = \rho \pi r^2$

$\frac{\delta m}{\delta r} = \rho 2\pi r$

$\delta m = \rho 2\pi r \delta r$

$A = $ length × width $= \pi a^2$.



∴ disc as a circle with boundary

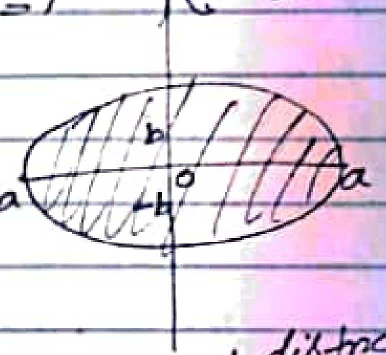
exP 48

find M.I of a uniform elliptical plate with semi major & minor axis about i) a major axis → i.e x-axis
ii) a minor axis.
iii) about an axis through the centre & ⊥ to the plate.

Sol:-

$\rho = \frac{M}{V} = \frac{M}{A}$
$M = \rho A$
$dM = \rho dA$

eq of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



here,

$I_{xx} = \int_A y^2 dm$

$= \iint y^2 \rho dA$

$= \int_{-a}^{a}\int_{-y}^{y} y^2 \rho \, dy \, dx$

⊥ distnce $= y^2$

$= \rho \int_{-a}^{a} \frac{y^3}{3}\Big|_{-y}^{y} dx = \frac{1}{3}\rho \int_{-a}^{a} (y^3 + y^3) dx$

$= \frac{1}{3}\rho \int_{-a}^{a} 2y^3 dx = \frac{2}{3}\rho \int_{-a}^{a} \frac{b^3}{a^3}(a^2 - x^2)^{3/2} dx$

$= \frac{2b^3}{3a^3}\rho \int_{-a}^{a} (a^2 - x^2)^{3/2} dx = \rho \frac{2b^3}{3a^3} \times 2 \int_{0}^{a} (a^2 - x^2)^{3/2} dx$

due to limit half

Put $x = a\sin\theta$
$dx = a\cos\theta \, d\theta$

at $x = 0$, $\theta = \sin^{-1} 0 = 0$
at $x = a$, $\theta = \sin^{-1} 1 = \pi/2$.

$a^2 - a^2\sin^2\theta$
$= (a^2\cos^2\theta)^{3/2}$
$= a^3\cos^3\theta$.

$= " " \int_0^{\pi/2} a^4 \cos^4\theta \, d\theta$.

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$
$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$
$y = \frac{b}{a}(a^2 - x^2)^{1/2}$
$y^3 = \frac{b^3}{a^3}(a^2 - x^2)^{3/2}$

$= \frac{4}{3}\rho \frac{b^3}{a^3} \int_0^{\pi/2} a^3\cos^3\theta \cdot a\cos\theta \, d\theta$