The Confluent Hypergeometric Function

(The Pochhammer - Barnes Hypergeometric Function)

It is defined as

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$, where

b is neither zero nor a negative integer and z is finite.

Theorem: If z is finite, then

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{zt}\, dt$$

Proof: Consider ${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b)}{\Gamma(b+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} B(a+n;b-a)$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \left[\int_0^1 t^{a+n}(1-t)^{b-a-1} dt\right] \frac{z^n}{n!}$$

Since the series is uniformly convergent $\therefore$

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-b}\left(\sum_{n=0} \frac{(tz)^n}{n!}\right)$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{tz}\, dt.$$

* L^P - boundedness of Integral (Proved) operators involving hypergeometric functions.

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Theorem: If z is finite then

$${}_mF_m\left(\frac{a}{m}, \frac{a+1}{m}, \ldots, \frac{a+m-1}{m}, \frac{b}{m}, \frac{b+1}{m}, \ldots \frac{b+m-1}{m}; z\right)$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{zt^n}\, dt.$$

Proof: ${}_mF_m\left(\frac{a}{m}, \frac{a+1}{m}, \ldots \frac{a+m-1}{m}, \frac{b}{m}, \frac{b+1}{m}, \ldots \frac{b+m-1}{m}; z\right)$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{a}{m}\right)_n \left(\frac{a+1}{m}\right)_n \cdots \left(\frac{a+m-1}{m}\right)_n}{\left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-1}{m}\right)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{mn}\, m^{-mn}}{(b)_{mn}\, m^{-mn}} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+mn)}{\Gamma(a)} \cdot \frac{\Gamma(b)}{\Gamma(b+mn)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+mn)\,\Gamma(b-a)}{\Gamma(b+mn)} \frac{z^n}{n!}$$

$\rightarrow B(a+mn, b-a)$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \sum_{n=0}^{\infty} \left(\int_0^1 t^{a+mn-1}(1-t)^{b-a-1} dt\right) \frac{z^n}{n!}$$

$\because$ Series is uniformly cgt.

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} \left(\sum_{n=0}^{\infty} \frac{t^{mn} z^n}{n!}\right) dt$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1}(1-t)^{b-a-1} e^{zt^m} dt$$

Hence Proved.

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The Confluent HyperGeometric Differential Equation

$$zw'' + (b-z)w' - aw = 0$$

Two singular Pts. $z = 0, \infty$

Kummer's First Formula:

If b is neither zero nor a negative integer, then

$$ {}_1F_1(a; b; z) = e^{z} {}_1F_1(b-a; b; -z)$$

Proof:

$$e^{-z} {}_1F_1(a; b; z) = \left(\sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{(a)_m}{(b)_m} \frac{z^m}{m!}\right)$$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^n (a)_m z^{n+m}}{(b)_m\, m!\, n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^{n-m} (a)_m z^n}{(b)_m\, m!\, (n-m)!} \qquad \because \frac{1}{(n-m)!} = \frac{(-1)^m (-n)_m}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m (-1)^{n-m} (-n)_m (a)_m z^n}{(b)_m\, n!\, m!}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(-n)_m (a)_m}{(b)_m} \frac{1}{m!}\right] \frac{(-z)^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_2F_1(-n, a; b; 1) \frac{(-z)^n}{n!}$$

$$\because F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(b-a)_n}{(b)_n} \frac{(-z)^n}{n!}$$

$$= {}_1F_1(b-a; b; -z)$$

Hence Proved.

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Derive the Confluent Hypergeometric Diff Equation

$$z w'' + (b - z) w' - a w = 0 \quad \text{—} \; (1)$$

where $w = {}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$ is the solution

Consider

$$\theta(\theta + b - 1) w = \theta(\theta + b - 1) \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{n(n+b-1)(a)_n}{(b)_n} \frac{z^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{(n+b-1)(a)_n}{(b)_n} \cdot \frac{z^n}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_{n-1}} \frac{z^n}{(n-1)!}$$

Replacing n by $n+1$.

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_n} \frac{z^{n+1}}{n!}$$

$$= z \sum_{n=0}^{\infty} \frac{(a+n)(a)_n}{(b)_n} \frac{z^n}{n!}$$

$$= z(a + \theta) w$$

$\Rightarrow$

$$[\theta(\theta + b - 1) - z(a + \theta)] w = 0$$

$$\theta^2 w + \theta b w - \theta w - a z w - z \theta w = 0$$

Since $\theta = z \frac{d}{dz}$

$\Rightarrow$ $\theta w = z w'$

$$\theta^2 w = z^2 w'' + z w'$$

$$z^2 w'' + \cancel{z w'} + b z w' - \cancel{z w'} - z^2 w' - a z w = 0$$

$$z^2 w'' + (b - z) z w' - a z w = 0$$

$$z w'' + (b - z) w' - a w = 0$$

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Conversely Let $w = \sum_{n=0}^{\infty} d_n z^n$ be the series solution of eq. (1)

Now

$$\frac{dw}{dz} = \sum_{n=1}^{\infty} n d_n z^{n-1}, \quad \frac{d^2w}{dz^2} = \sum_{n=2}^{\infty} n(n-1) d_n z^{n-2}$$

(1) $\Rightarrow$

$$z \sum_{n=2}^{\infty} n(n-1) d_n z^{n-2} + (b-z) \sum_{n=1}^{\infty} n d_n z^{n-1} - a \sum_{n=0}^{\infty} d_n z^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) d_n z^{n-1} + b \sum_{n=1}^{\infty} n d_n z^{n-1} - \sum_{n=1}^{\infty} n d_n z^n$$
$$- a \sum_{n=0}^{\infty} d_n z^n = 0.$$

Replace n by $n+1$ in 1^{st} & 2^{nd} Term

$$\sum_{n=1}^{\infty} n(n+1) d_{n+1} z^n + b \sum_{n=0}^{\infty} (n+1) d_{n+1} z^n$$
$$- \sum_{n=1}^{\infty} n d_n z^n - a \sum_{n=0}^{\infty} d_n z^n = 0$$

Comparing coefficients of z^n.

$$n(n+1) d_{n+1} + b(n+1) d_{n+1} - n d_n - a d_n = 0$$

$$\Rightarrow \quad (n+b)(n+1) d_{n+1} = (a+n) d_n$$

$$\Rightarrow \quad d_{n+1} = \frac{a+n}{(n+1)(b+n)} d_n$$

Put $n = 0, 1, 2, \ldots$

$$d_1 = \frac{a}{(1)(b)} d_0 = \frac{a}{b} d_0$$

$$d_2 = \frac{(a+1)}{(b+1)2} d_1 = \frac{a(a+1)}{2b(b+1)} d_0 = \frac{(a)_2}{2(b)_2} d_0$$

$$d_3 = \frac{(a+2)}{3(b+2)} d_2 = \cdots = \frac{(a)_3}{3!(b)_3} d_0$$

$$d_n = \frac{(a)_n}{(b)_n n!} d_0$$

So $w = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} d_0 \cdot z^n$

For convience we take $d_0 = 1$

So

$$w = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{z^n}{n!}$$ is the required

Sol. of (1).

Incomplete Beta Function

$$B(\alpha, \beta) = \int_0^x t^{\alpha-1}(x-t)^{\beta-1} dt$$

Incomplete Gamma Function

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$$

Theorem: If $Re(\alpha) > 0$, $Re(\beta) > 0$, If m is a Positive integer, then inside the region of convergence of resultant series

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} {}_pF_q \left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix} cx^m\right] dx.$$

$$= B(\alpha, \beta)\, t^{\alpha+\beta-1} {}_{p+m}F_{q+m} \left[\begin{matrix} a_1, a_2, \ldots, a_p, \frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots, \frac{\alpha+m-1}{m}; \\ b_1, b_2, \ldots, b_q, \frac{\alpha+\beta}{m}, \frac{\alpha+\beta+1}{m}, \ldots, \frac{\alpha+\beta+m-1}{m}; \end{matrix} ct^m\right]$$

Proof: Consider

$${}_{p+m}F_{q+m} \left[\begin{matrix} a_1, a_2, \ldots, a_p, \frac{\alpha}{m}, \frac{\alpha+1}{m}, \ldots \frac{\alpha+m-1}{m} \\ b_1, b_2, \ldots b_q, \frac{\alpha+\beta}{m}, \ldots \frac{\alpha+\beta+m-1}{m}, \end{matrix} ct^m\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{\left(\frac{\alpha}{m}\right)_n \left(\frac{\alpha+1}{m}\right)_n \cdots \left(\frac{\alpha+m-1}{m}\right)_n}{\left(\frac{\alpha+\beta}{m}\right)_n \left(\frac{\alpha+\beta+1}{m}\right)_n \cdots \left(\frac{\alpha+\beta+m-1}{m}\right)_n} \frac{(ct^m)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{(\alpha)_{mn}}{(\alpha+\beta)_{mn}} \frac{(ct^m)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \cdot \frac{\Gamma(\alpha+mn)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+mn)} \frac{c^n t^{mn}}{n!}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \cdot \frac{\Gamma(\alpha+mn)\Gamma(\beta)}{\Gamma(\alpha+\beta+mn)} \frac{c^n t^{mn}}{n!}$$

$$= \frac{1}{B(\alpha,\beta)} \cdot \sum_{m=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \int_0^1 s^{\alpha+mn-1} (1-s)^{\beta-1} \frac{c^n t^{mn}}{n!} ds$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 s^{\alpha-1} (1-s)^{\beta-1} \left(\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \cdot \frac{(ct^m s^m)^n}{n!} \right) ds$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 s^{\alpha-1} (1-s)^{\beta-1} {}_pF_q \left[\begin{matrix} a_1, \ldots a_p ; \\ b_1, \ldots b_q ; \end{matrix} \, c t^m s^m \right] ds$$

Put $ts = x \Rightarrow s = \frac{x}{t}$

$s = 0 \Rightarrow x = 0$

$s = 1 \Rightarrow x = t$

$$= \frac{1}{B(\alpha,\beta)} \int_0^t \left(\frac{x}{t}\right)^{\alpha-1} \left(1-\frac{x}{t}\right)^{\beta-1} {}_pF_q \left[\begin{matrix} a_1, \ldots a_p \\ b_1, \ldots b_q \end{matrix} ; c x^m \right] \frac{dx}{t}$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^t x^{\alpha-1} (t-x) \cdot \frac{1}{t^{\alpha+\beta-2+1}} {}_pF_q \left[\begin{matrix} a_1, \ldots a_p \\ b_1, \ldots b_q \end{matrix} , c x^m \right] dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^t x^{\alpha-1} (t-x)^{\beta-1} \frac{1}{t^{\alpha+\beta-1}} {}_pF_q [\,'' \, ; c x^m] dx$$

Pochhammer's k - Symbol:

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k), & n \geq 1, k>0 \\ 1 & , n=0 \end{cases}$$

Gamma k - Function:

$$\Gamma_k(z) = \int_0^\infty e^{-t^k/k}\, t^{z-1}\, dt, \quad Re(z)>0, k>0$$

Beta k - Function

$$B_k(\beta, \alpha) = \frac{1}{k}\int_0^1 t^{\beta/k - 1}(1-t)^{\alpha/k - 1}\, dt$$

Diff. Eq.

$$kx(1-kx)y'' + [c-(a+b+k)kx]y' - aby = 0$$

$$_2F_{1,k}\left[\begin{matrix}(a,k),(b,k);\\(c,k);\end{matrix}\, x\right] = \sum_{n=0}^\infty \frac{(a)_{n,k}\,(b)_{n,k}}{(c)_{n,k}}\frac{x^n}{n!}$$

Confluent k - hypergeometric function.

$$_1F_{1,k}\left[\begin{matrix}(a,k);\\(b,k);\end{matrix}\, x\right] = \sum_{n=0}^\infty \frac{(a)_{n,k}}{(b)_{n,k}}\frac{x^n}{n!}$$

$$I^\alpha(f(x)) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

$$I_k^\alpha(f(x)) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\, dt$$

Examples. Pochhammer's k Symbol.

$$(5)_{3,4} = 5(5+4)(5+8)$$
$$= 5 \cdot 9 \cdot 13$$

$$(4)_{4,4} = 4 \cdot 8 \cdot 12 \cdot 16$$

Pochhamma Symbol

$$(5)_3 = 5(5+1)(5+2)$$
$$= 5 \cdot 6 \cdot 7$$

$$(\sqrt{2})_{4,\sqrt{3}} = (\sqrt{2})(\sqrt{2}+\sqrt{3})(\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{3})$$

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Exercise Chapter 07

Problem(3) Prove that

$$(b)_k \frac{d^k}{dz^k}\left[e^{-z}\,{}_1F_1(a;b;z)\right] = (-1)^k (b-a)_k\, e^{-z} F(a;b+k;z)$$

Solution:

$$\frac{d}{dz}\left[{}_1F_1(a;b;z)\right] = \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^{n-1}}{(n-1)!}$$

Replac n by n+1

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(b)_{n+1}} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{a(a+1)_n}{b(b+1)_n} \frac{z^n}{n!}$$

$$= \frac{a}{b}\,{}_1F_1\left[\begin{matrix}a+1; \\ b+1;\end{matrix}\ z\right] \quad -(1)$$

Also

$$\frac{d}{dz}\left[{}_1F_1(b-a;b;-z)\right] = \sum_{n=1}^{\infty} \frac{(b-a)_n (-1)^n z^{n-1}}{(b)_n (n-1)!}$$

from (1) we have

$$\frac{d^2}{dz^2} F(a;b;z) = \frac{d}{dz}\left[\frac{a}{b} F\left(\begin{matrix}a+1 ; \\ b+1 ;\end{matrix}\ z\right)\right]$$

$$= \frac{a}{b}\left[\frac{a+1}{b+1}\,{}_1F_1\left(\begin{matrix}a+2 ; \\ b+2 ;\end{matrix}\ z\right)\right]$$

$$= \frac{(a)_2}{(b)_2}\,{}_1F_1\left(\begin{matrix}a+2 ; \\ b+2 ;\end{matrix}\ z\right)$$

Continuing in the same way

$$\frac{d^k}{dz^k} \quad \frac{(a)_k}{(b)_k}\,{}_1F_1\left(\begin{matrix}a+k ; \\ b+k ;\end{matrix}\ z\right)$$

Since by Kummer's 1st formula

$$e^{-z}\,{}_1F_1[a;b;z] = F[b-a;b;-z]$$

$$\frac{d^k}{dz^k}\left[e^{-z}\,{}_1F_1(a;b;z)\right] = \frac{d^k}{dz^k}\left[b-a;b;-z\right]$$

$$= \frac{(-1)^k (b-a)_k}{(b)_k}\,{}_1F_1(b-a+k;b+k;-z)$$

$\because \quad {}_1F_1(a;b;z) = e^{z}\,{}_1F_1(b-a;b;-z)$

$\Rightarrow$

$$= \frac{(-1)^k (b-a)_k}{(b)_k} e^{-z}\,{}_1F_1(a;b+k;z)$$

$= R.H.S.$

Question 4: Show that

$${}_1F_1(a;b;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1}\,{}_0F_1(-;b;zt)\,dt$$

Solution: ${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$

$$= \frac{1}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\, z^n}{(b)_n\, n!}$$

$\because \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$

So $\quad \Gamma(a+n) = \int_0^\infty e^{-t} t^{a+n-1}\,dt.$

$$= \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} \sum_{n=0}^{\infty} t^{a+n-1} \frac{z^n}{(b)_n\, n!}$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \sum_{n=0}^{\infty} \frac{z^n t^n}{(b)_n\, n!}\,dt$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1}\,{}_0F_1(-;b;zt)\,dt$$

Question 5: Show that with the aid of the result in exe 4.

$$\int_0^\infty \exp(-t^2)\, t^{2a-n-1} J_n(zt)\,dt = \frac{\Gamma(a)\, z^n}{2^{n+1}\Gamma(n+1)}\,{}_1F_1\left(a, n+1, \frac{-z^2}{4}\right)$$

Solution: L.H.S.

$$\int_0^\infty \exp(-t^2)\, t^{2a-n-1} \frac{\left(\frac{zt}{2}\right)^n}{\Gamma(1+n)}\,{}_0F_1\left(-, 1+n, \frac{-(zt)^2}{4}\right)dt$$

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Put $t^2 = \beta \Rightarrow 2t\,dt = 2\beta$

$\Rightarrow t = \beta^{1/2}$

$$= \int_0^\infty e^{-\beta} \frac{\beta^{1/2(2a-n-1)} z^n \beta^{n/2}}{2^n \Gamma(1+n)} {}_0F_1\left(-;1+n;-\frac{z^2\beta}{4}\right)\frac{d\beta}{2\beta^{1/2}}$$

$$= \int_0^\infty e^{-\beta} \frac{\beta^{a-1} z^n}{2^{n+1}\Gamma(1+n)} {}_0F_1\left(-,1+n,-\frac{z^2\beta}{4}\right) d\beta.$$

$$= \frac{z^n}{2^{n+1}\Gamma(1+n)} \int_0^\infty e^{-\beta}\beta^{a-1} {}_0F_1\left(-,1+n,-z^2\frac{\beta}{4}\right)d\beta$$

$$= \frac{z^n \cancel{\Gamma(a)}}{2^{n+1}\cancel{\Gamma(1+n)}} {}_0F_1\left(-;\cancel{1+n};-\cancel{z^2}\frac{\beta}{4}\right)$$

$$= \frac{z^n}{2^{n+1}\Gamma(1+n)} \int_0^\infty e^{-\beta}\beta^{a-1}\left(\sum_{m=0}^\infty \frac{(-1)^m z^{2m}\beta^m}{(1+n)_m (4)^m}\cdot\frac{1}{m!}\right)d\beta$$

$$= \frac{z^n}{2^{n+1}\Gamma(1+n)} \sum_{m=0}^\infty \int_0^\infty e^{-\beta}\beta^{a+m-1}\,d\beta\ \frac{(-1)^m z^{2m}}{(1+n)_m m! (4)^m}$$

$$= \frac{z^n}{2^{n+1}\Gamma(1+n)} \sum_{m=0}^\infty \frac{\Gamma(a+m)(-1)^m z^{2m}}{(1+n)_m\, m!\, 4^m}\cdot\frac{\Gamma(a)}{\Gamma(a)}$$

$$= \frac{z^n\Gamma(a)}{2^{n+1}\Gamma(1+n)} \sum_{m=0}^\infty \frac{(a)_m (-1)^m z^{2m}}{(1+n)_m\, m!\, 4^m}$$

$$= \frac{z\Gamma(a)}{2^{n+1}\Gamma(1+n)} {}_1F_1\left(a;1+n;-\frac{z^2}{4}\right)$$

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Question 06: If k and n are non-negative integers, then show that:

$$F\left[\begin{matrix}-k, & a+n; \\ & a; \end{matrix} 1\right] = \begin{cases} 0 & \text{for } k > n \\ \frac{(-n)_k}{(a)_k} & \text{for } 0 \le k \le n \end{cases}$$

Solution: Let $V(k,n) = F\left[\begin{matrix}-k; & a+n; \\ a; \end{matrix} 1\right]$

$$= \sum_{m=0}^{k} \frac{(-k)_m (\alpha+n)_m}{(a)_m \, m!}$$

$\because k > n$
F_0

$\because (k-m)! = (-1)^m \, k!$
$\Rightarrow (-k)_m = \frac{(-1)^m k!}{(k-m)!}$

$$= \sum_{m=0}^{k} \frac{(-1)^m k! (a+n)_m (a)_n}{(a)_m (a)_n (k-n)! \, m!}$$

$$= \sum_{m=0}^{n} \frac{(-1)^m k! (\alpha+n)_m}{(a)_m (k-m)! \, m!}$$

Let

$$\psi = \sum_{k=0}^{\infty} \frac{V(k,n)}{k!} t^k$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^m (\alpha+n)_m k!}{(a)_m (k-m)! \, m!} \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (\alpha+n)_m \; t^{m+k}}{(a)_m \, k! \, m!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{(-1)^m (\alpha+n)_m t^m}{(a)_m \, m!}\right)$$

$$= e^t F(\alpha+n; a; -t)$$

$$= {}_1F_1(-n; \alpha; t) \quad \text{Kummer's 1st formula}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k t^k}{(\alpha)_k k!} + 0$$

$$= \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha)_k} \frac{t^k}{k!} + \sum_{k=n+1}^{\infty} \frac{(-n)_k t^k}{(\alpha)_k k!}$$

$$= \frac{(-n)_k}{(\alpha)_k} \quad \text{if } 0 \le k \le n$$

0 if $k > n$

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