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Bessel Function:

The Bessel function of 1st kind is defined as

$$J_n(z) = \frac{(z/2)^n}{\Gamma(1+n)} \, {}_0F_1\left(-; 1+n, -\frac{z^2}{4}\right) \quad \text{—} (1)$$

where n is not a negative integer.

If n is a negative integer, then

$$J_n(z) = (-1)^n J_{-n}(z)$$

Equation (1) may be written as.

$$J_n(z) = \frac{(z/2)^n}{\Gamma(1+n)} \, {}_0F_1\left(-, 1+n; -\frac{z^2}{4}\right)$$

$$= \frac{(z/2)^n}{\Gamma(1+n)} \sum_{m=0}^{\infty} \frac{1}{(1+n)_m} \cdot \frac{(-1)^m z^{2m}}{m! \cdot 2^{2m}}$$

$$= \frac{1}{\Gamma(1+n)} \sum_{m=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(1+n+m)} \cdot \frac{(-1)^m z^{2m+n}}{m! \, 2^{2m+n}}$$

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n} m! \Gamma(1+n+m)}$$

If we Replace z by $-z$ then

$$J_n(-z) = (-1)^n J_n(z)$$

Bessel's Differential Equation:

$$[\theta(\theta+b-1) - y] \, u = 0, \quad (1) \quad \theta = y\frac{d}{dy}$$

Let $b = 1+n$, $y = -z^2/4$

whose $u = {}_0F_1(-; b, y)$ is a. Solution of eq. (1).

Now let $b = 1+n$, $y = -z^2/4$ Put in eq. (1)

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$zU'' + (2n+1)U' + zU = 0$ where $u = {}_0F_1\left(-; 1+n; -\frac{z^2}{4}\right)$

Now let $w = z^n u$

$\Rightarrow \quad U = z^{-n} w$

Then we have the following differential eq.

$z^2 w'' + z w' + (z^2 - n^2) w = 0$, — (2)

where $w = z^n {}_0F_1\left(-; 1+n; -z^2/4\right)$ is a solution of eq. (2) which is called Bessel differential equation.

$b = 1+n$

$y = -z^2/4$

$$\theta(\theta+b-1)u = \theta(\theta+b-1)\sum_{m=0}^{\infty} \frac{1}{(b)_m} \cancel{(-1)^m} \frac{y^m}{m!}$$

$$= \sum_{m=0}^{\infty} m(m+b-1) \frac{1}{(b)_m} \cancel{(-1)^m} \frac{y^m}{m!}$$

$$= \sum_{m=1}^{\infty} (m+b-1) \frac{1}{(b)_m} \cancel{(-1)^m} \frac{y^m}{(m-1)!}$$

~~Replace m by m+1~~

$$\theta(\theta+b-1)u = \sum_{m=1}^{\infty} \frac{1}{(b)_{m-1}} \cdot \frac{y^m \cancel{(-1)^m}}{(m-1)!}$$

Replace m by $m+1$

$$\theta(\theta+b-1)u = \sum_{m=0}^{\infty} \frac{1}{(b)_m} \frac{y^{m+1} \cancel{(-1)^{m+1}}}{m!}$$

$$\theta(\theta+b-1)U = y\sum_{m=0}^{\infty} \frac{1}{(b)_m} \frac{y^m}{m!}$$

$\Rightarrow \quad \theta(\theta+b-1)U - yU = 0$

$\Rightarrow \quad [\theta(\theta+b-1) - y]U = 0$ — (1)

$\theta^2 U + \theta b U - \theta U - yU = 0$

$\because \quad \theta U = y\frac{dU}{dy} = yU'$

$\theta^2 U = \theta(yU') = y\frac{d}{dy}(yU')$

$\theta^2 U = yU' + y^2 U''$

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$$yv' + y^2 v'' + byv' - yv' - yv = 0$$

$$y^2 v'' + byv' - yv = 0$$

$$\Rightarrow \quad yv'' + bv' - v = 0 \quad \text{—— (2)}$$

Since $y = -\frac{z^2}{4}$, $b = 1+n$.

$$\frac{du}{dy} = \frac{dv}{dz}\frac{dz}{dy}$$

$$\frac{dv}{dy} = -\frac{2}{z}\frac{dv}{dz}$$

$$\frac{d^2v}{dy} = \frac{d}{dz}\left[-\frac{2}{z}\frac{dv}{dz}\right]\frac{dz}{dy}$$

$$= \left[\frac{d}{dz}\left(-\frac{2}{z}\right)\frac{du}{dz} + \left(-\frac{2}{z}\right)\frac{d}{dz}\left(\frac{dv}{dz}\right)\right]\left(-\frac{2}{z}\right)$$

$$= \left(\frac{2}{z^2}\frac{dv}{dz} - \frac{2}{z}\frac{d^2v}{dz^2}\right)\left(-\frac{2}{z}\right)$$

$$\frac{d^2v}{dy^2} = -\frac{4}{z^3}\frac{du}{dz} + \frac{4}{z^2}\frac{d^2v}{dz^2}$$

using values of y, v', v'', and b in (2)

$$-\frac{z^2}{4}\left[-\frac{4}{z^3}\frac{du}{dz} + \frac{4}{z^2}\frac{d^2v}{dz^2}\right] + (1+n)\left(-\frac{2}{z}\right)\frac{du}{dz} - u = 0$$

$$\frac{1}{z}\frac{du}{dz} \mp \frac{d^2v}{dz^2} + (1+n)\left(-\frac{2}{z}\right)\frac{du}{dz} - v = 0$$

$$\frac{1}{z}\frac{dv}{dz} - \frac{2}{z}\frac{dv}{dz} - \frac{2n}{z}\frac{du}{dz} \mp \frac{d^2v}{dz^2} - v = 0$$

$$-\frac{1}{z}(1+2n)\frac{dv}{dz} - \frac{d^2u}{dz^2} - v = 0$$

$$\Rightarrow \quad z\frac{d^2v}{dz^2} + (2n+1)\frac{dv}{dz} + zv = 0$$

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$$\Rightarrow \quad z\frac{d^2U}{dz^2} + (2n+1)\frac{dU}{dz} + zU = 0 \quad \text{—— (3)}$$

Now Put $\omega = z^n U$

$$\Rightarrow \quad U = z^{-n}\omega$$

$$\Rightarrow \quad \frac{dU}{dz} = -n z^{-n-1}\omega + z^{-n}\frac{d\omega}{dz}$$

$$\frac{dU}{dz} = -n z^{-n-1}\omega + z^{-n}\frac{d\omega}{dz}$$

Now

$$\frac{d^2U}{dz^2} = (-n)(-n-1)z^{-n-2}\omega - n z^{-n-1}\frac{d\omega}{dz} - n z^{-n-1}\frac{d\omega}{dz} + z^{-n}\frac{d^2\omega}{dz^2}$$

$$\frac{d^2U}{dz^2} = z^{-n-2} n(n+1)w - 2n z^{-n-1}\frac{d\omega}{dz} + z^{-n}\frac{d^2\omega}{dz^2}$$

using these values in eq. (3), we have

$$z\left[\begin{array}{l} n(n+1) z^{-n-2} w \\ -2n z^{-n-1}\frac{dw}{dz} + z^{-n}\frac{d^2w}{dz^2}\end{array}\right] + (2n+1)\left(-n z^{-n-1}\omega + z^{-n}\frac{d\omega}{dz}\right)$$

$$+ z \cdot z^{-n}\omega = 0$$

$$n(n+1)w z^{-n-1} - 2n z^{-n}\frac{d\omega}{dz} + z^{-n+1}\frac{d^2\omega}{dz^2}$$
$$- n(2n+1) z^{-n-1}\omega + (2n+1) z^{-n}\frac{d\omega}{dz} + z^{-n+1}\omega = 0$$

$$\Rightarrow -n^2 w z^{-n-1} + z^{-n}\frac{dw}{dz}$$

$\therefore n^2 + n - 2n^2 - n = -n^2$

$$+ z^{-n+1}\frac{d^2w}{dz^2} + z^{-n+1}\omega = 0$$

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$$z^{-n-1}\left[-n^2 w + z\frac{dw}{dz} + z^2\frac{d^2w}{dz^2} + z^2 w\right] = 0$$

Since $z^{-n-1} \neq 0$

So

$$-n^2 w + z\frac{dw}{dz} + z^2\frac{d^2w}{dz^2} + z^2 w = 0$$

$$\Rightarrow \quad (z^2 - n^2)w + z\frac{dw}{dz} + z^2\frac{d^2w}{dz^2} = 0$$

let

$$\frac{dw}{dz} = w^{\cdot}$$

$$\frac{d^2w}{dz^2} = w^{\cdot\cdot}$$

$\Rightarrow$

$$z^2 w^{\cdot\cdot} + z w^{\cdot} + (z^2 - n^2)w = 0$$

its required.

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Differential Recurrence Relation:

Consider the Bessel function of 1st kind

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n} \Gamma(1+n+m)\, m!}$$

$$\frac{d}{dz}\left[z^n J_n(z)\right] = \frac{d}{dz}\left[\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+2n}}{2^{2m+n}\Gamma(1+n+m)\, m!}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n) z^{2m+2n-1}}{2^{2m+n}(m+n)\Gamma(m+n)\, m!} \qquad \because \Gamma(m+n+1) = (m+n)\Gamma(m+n)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot z^{2m+2n-1}}{2^{2m+n-1}\Gamma(n+m)\, m!}$$

$\Rightarrow$

$$\frac{d}{dz}\left[z^n J_n(z)\right] = z^n \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n-1}}{2^{2m+n-1}\Gamma(1+\overline{n-1}+m)\, m!}$$

$$z^n J_n'(z) + nz^{n-1} J_n(z) = z^n J_{n-1}(z)$$

$$\Rightarrow z^{n-1}\left[z J_n'(z) + n J_n(z)\right] = \left[z J_{n-1}(z)\right] z^{n-1}$$

$\Rightarrow$

$$z J_n'(z) + n J_n(z) = z J_{n-1}(z)$$

$$z J_n'(z) = z J_{n-1}(z) - n J_n(z) \quad \text{—— (1)}$$

Again multiplying by z^{-n}

$$\frac{d}{dz}\left[z^{-n} J_n(z)\right] = \frac{d}{dz}\left[\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m+n}\Gamma(1+n+m)\, m!}\right]$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m\, 2m\, z^{2m-1}}{2^{2m+n}\Gamma(1+n+m)\, m!} = \sum_{m=1}^{\infty} \frac{(-1)^m z^{2m-1}}{2}$$

Shift the index m by $m+1$.

$$\frac{d}{dz}\left[z^{-n} J_n(z)\right] = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, 2(m+1) \cdot z^{2m+1}}{2^{2m+n+1}\Gamma(1+n+m+1)\,(m+1)!}$$

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$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \cdot z^{2m+1}}{2^{2m+n+1}\Gamma(1+\overline{1+n+m})\, m!}$$

$$= - z^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n+1}}{2^{2m+n+1}\Gamma(1+\overline{n+1}+m)\, m!}$$

$\Rightarrow$

$$\frac{d}{dz}\left[z^{-n} J_n(z)\right] = - z^{-n} J_{n+1}(z)$$

$$z^{-n} J_n'(z) - n z^{-n-1} J_n(z) = - z^{-n} J_{n+1}(z)$$

$\Rightarrow$

$$z J_n'(z) - n J_n(z) = - z J_{n+1}(z)$$

$$\Rightarrow \quad z J_n'(z) = n J_n(z) - z J_{n+1}(z) \quad —(2)$$

Adding Eqs. (1) & (2)

$$2z J_n'(z) = z J_{n-1}(z) - z J_{n+1}(z)$$

or

$$2 J_n'(z) = J_{n-1}(z) - J_{n+1}(z) \quad —(3)$$

Pure Recurrence Relation for Bessel Function:

(1) − (2) $\Rightarrow$

$$0 = z J_{n-1}(z) - n J_n(z) - n J_n(z) + z J_{n+1}(z)$$

$$z J_{n-1}(z) - 2n J_n(z) + z J_{n+1}(z) = 0 \quad —(4)$$

It can be written as replacing n by $n-1$

$$z J_{n-2}(z) - 2(n-1) J_{n-1}(z) + z J_n(z) = 0$$

$$\Rightarrow \quad J_n(z) = \frac{2(n-1)}{z} J_{n-1}(z) - J_{n-2}(z)$$

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Lemma: If $n \geq 1$ then

$$\sum_{m=0}^{n} A(m,n) = \sum_{m=0}^{[n/2]} A(m,n) + \sum_{m=0}^{[\frac{n-1}{2}]} A(n-m,n)$$

Proof: For $n \geq 1$, $n = 1 + [\frac{n}{2}] + [\frac{n-1}{2}]$

$$\sum_{m=0}^{n} = \sum_{m=0}^{[\frac{n}{2}]} A(m,n) + \sum_{m=[\frac{n}{2}]+1}^{1+[\frac{n}{2}]+[\frac{n-1}{2}]} A(m,n)$$

Replace m by $n-m$ i.e. m by $\underline{1+[\frac{n}{2}]+[\frac{n-1}{2}]-m}$ in 2nd ser
n

$$= \sum_{m=0}^{[\frac{n}{2}]} A(m,n) + \sum_{m=[\frac{n-1}{2}]}^{0} A(n-m,n)$$

$$= \sum_{m=0}^{[n/2]} A(m,n) + \sum_{m=0}^{[\frac{n-1}{2}]} A(n-m,n)$$

Theorem: If $t \neq 0$ and $\forall$ finite z, then

$$\exp\left[\frac{1}{2} z\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$$

Proof: Consider $\sum_{n=-\infty}^{\infty} J_n(z)\, t^n = \sum_{n=-\infty}^{-1} J_n(z) t^n + \sum_{n=0}^{\infty} J_n(z)\, t^n$

Replace n by $-n-1$ in 2nd Σ

$$= \sum_{n=0}^{\infty} J_{-n-1}(z) t^{-n-1} + \sum_{n=0}^{\infty} J_n(z) t^n$$

$-n-1 = -1$
$n = 0$
$-n-1 = -\infty$
$n = \infty$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} J_{n+1}(z)\, t^{-n-1} + \sum_{n=0}^{\infty} J_n(z)\, t^n \quad (1)$$

$\because J_{-n}(z) = (-1)^n J_n(z)$

Now using $J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n}\,\Gamma(1+n+m)\, m!}$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n+1} (-1)^{n+1}}{2^{2m+n+1}\,\Gamma(1+n+1+m)\, m!}\, t^{-n-1} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n}\,\Gamma(1+n+m)\, m!}\, t^n$$

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$$= \sum_{n,m=0}^{\infty} \frac{(-1)^{m+n+1} \cdot z^{2m+n+1} t^{-n-1}}{2^{2m+n+1}\Gamma(n+2+m)\, m!} + \sum_{m,n=0}^{\infty} \frac{(-1)^m z^{2m+n} t^n}{2^{2m+n}\Gamma(1+n+m)\, m!}$$

Now using Lemma.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} A(m, n-2m)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} \frac{(-1)^{n-m+1} z^{n+1} t^{-n+2m-1}}{2^{n+1}\Gamma(n+2-m)\, m!} + \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} \frac{(-1)^m z^n t^{n-2m}}{2^n \Gamma(1+n-m)\, m!}$$

Replacing ~~n~~

$$= \sum_{n=1}^{\infty}\sum_{m=0}^{[\frac{n-1}{2}]} \frac{(-1)^{n-m} t^{-n+2m}}{\Gamma(n+1-m)\, m!} \frac{z^n}{2^n} + 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{[n/2]} \frac{(-1)^m t^{n-2m}}{\Gamma(1+n-m)\, m!} \frac{z^n}{2^n}$$

Now using Lemma

$$\sum_{m=0}^{n} A(m,n) = \sum_{m=0}^{[n/2]} A(m,n) + \sum_{m=0}^{[\frac{n-1}{2}]} A(n-m, n)$$

$$= 1 + \sum_{n=1}^{\infty}\left\{ \sum_{m=0}^{[n/2]} \frac{(-1)^m t^{n-m-m}}{\Gamma(n+1-m)\, m!} + \sum_{m=0}^{[\frac{n-1}{2}]} \frac{(-1)^{n-m} t^{-n+m+m}}{\Gamma(1+n-m)\, m!} \right\} \frac{z^n}{2^n}$$

$$= 1 + \sum_{n=1}^{\infty}\left\{ \sum_{m=0}^{[n/2]} \frac{(-1)^m t^{n-m-m}}{(n-m)!\, m!} + \sum_{m=0}^{[\frac{n-1}{2}]} \frac{(-1)^{n-m} t^{m-(n-m)}}{(n-m)!\, m!} \right\} \frac{z^n}{2^n}$$

$$= 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m t^{n-m-m}}{(n-m)!\, m!} \frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m t^{n-m-m}}{(n-m)!\, m!} \frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \left(\frac{n!\, t^{n-m} (-t^{-1})^m}{n!\, (n-m)!\, m!} \right) \frac{z^n}{2^n}.$$

$$= \sum_{n=0}^{\infty} \frac{(t-1/t)^n}{n!} \frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{[\frac{z}{2}(t - 1/t)]^n}{n!}$$

$$= \exp\left[\frac{z}{2}(t - 1/t)\right]$$

Index Half an odd Integer.

Since

$$\sin z = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}$$

Since $(\alpha)_{2m} = (\alpha)(\alpha+1) \cdots (\alpha + 2m - 1)$

$\alpha = 2$

$(2)_{2m} = 2(3) \cdots (2m+1) = (2m+1)!$

$(2m+1)! = (2)_{2m}$

and $(2)_{2m} = 2^{2m} \left(\frac{2}{2}\right)_m \left(\frac{2+1}{2}\right)_m = 2^{2m} (1)_m \left(\frac{3}{2}\right)_m$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{2^{2m} m! (3/2)_m} = z \sum_{m=0}^{\infty} \frac{(-1)^m (z^2/4)^m}{m! (3/2)_m}$$

$$= z\, {}_0F_1(-;\ 3/2;\ -1/4\, z^2)$$

Since

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+n}}{2^{2m+n} m! \Gamma(1+n+m)}$$

Put $n = 1/2$

$$J_{1/2}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1/2}}{2^{2m+1/2} m! \Gamma(1 + 1/2 + m)}$$

$$= \sum_{m=0}^{\infty} \frac{z^{2m+1/2} (-1)^m}{2^{2m+1/2} m! \Gamma(3/2 + m)} \qquad \because (3/2)_m = \frac{\Gamma(3/2 + m)}{\Gamma(3/2)}$$

$$= \sum_{m=0}^{\infty} \frac{z^{2m+1/2} (-1)^m}{2^{2m+1/2} m! (3/2)_m \Gamma(3/2)}$$

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$$= \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{1/2} \frac{(-1)^m (z^2/4)^m}{m!\,(3/2)_m} \cdot \frac{1}{\Gamma(3/2)}$$

$$= \frac{(z/2)^{1/2}}{\Gamma(3/2)} \, {}_0F_1\left(-;\ 3/2;\ -\tfrac{1}{4}z^2\right)$$

and $\Gamma(3/2) = \Gamma(1+1/2) = 1/2\,\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$

$$J_{1/2}(z) = \left(\frac{2z}{\pi z}\right)^{1/2} \cdot \frac{\sin z}{z} = \sqrt{\frac{z}{2}} \Big/ \frac{1}{2}\sqrt{\pi} \cdot \frac{\sin z}{z}$$

$$= \sqrt{\frac{2}{\pi z}}\, \sin z$$

Prove that $J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z$.

Since $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

Since $(2n)! = (2n)(2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1$

$$= (2n+1-1)(2n-1) \cdots 3 \cdot 2 \cdot 1$$

$$= (1)_{2n}$$

Since $(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$

$\alpha = 1$

$$(1)_{2n} = 2^{2n} (1/2)_n (1)_n = 2^{2n} n!\, (1/2)_n$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^{2n} n!\, (1/2)_n}$$

$$\cos z = {}_0F_1\left(-;\ 1/2;\ -z^2/4\right)$$

Now

$$J_{-1/2}(z) = \frac{(z/2)^{-1/2}}{\Gamma(1/2)} \, {}_0F_1\left(-;\ 1/2;\ -z^2/4\right)$$

$$= \frac{(z/2)^{-1/2}}{\sqrt{\pi}} \cos z$$

$$= \left(\frac{2}{\pi z}\right)^{1/2} \cos z$$