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Exercise CHAPTER 5

Q. No. 3) with the aid of theorem 8, Show that
$$\frac{\Gamma(1+\tfrac{1}{2}a)}{\Gamma(1+a)} = \frac{\cos \tfrac{1}{2}\pi a\,\Gamma(1-a)}{\Gamma(1-\tfrac{1}{2}a)}$$
and that
$$\frac{\Gamma(1+a-b)}{\Gamma(1+\tfrac{1}{2}a-b)} = \frac{\sin \pi(b-\tfrac{1}{2}a)\,\Gamma(b-\tfrac{1}{2}a)}{\sin \pi(b-a)\,\Gamma(b-a)}$$
Thus Put Dixon's theorem in the form
$$_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b; 1+a-c; \end{matrix} \; 1\right] = \frac{\cos \tfrac{1}{2}\pi a \sin\pi(b-\tfrac{1}{2}a)}{\sin \pi(b-a)} \cdot$$
$$\frac{\Gamma(1-a)\Gamma(b-\tfrac{1}{2}a)\Gamma(1+a-c)\Gamma(1+\tfrac{1}{2}a-b-c)}{\Gamma(1-\tfrac{1}{2}a)\Gamma(b-a)\Gamma(1+\tfrac{1}{2}a-c)\Gamma(1+a-b-c)}$$

Solution: since $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}$ & $\Gamma(1+z) = z\Gamma(z)$

Now
$$\frac{\Gamma(1+a/2)\Gamma(1-a/2)}{\Gamma(1+a)\Gamma(1-a)} = \frac{a/2\,\Gamma(a/2)\Gamma(1-a/2)}{a\,\Gamma(a)\Gamma(1-a)}$$
$$= \frac{a/2 \quad \pi/\sin\pi(a/2)}{a \cdot \pi/\sin\pi a}$$
$$= \frac{a/2 \cdot \pi}{\sin\frac{\pi a}{2}} \cdot \frac{\sin\pi a}{\pi a} = \frac{\sin\pi a}{2\sin\frac{\pi a}{2}}$$
$$= \frac{2\sin\pi a/2 \cdot \cos\pi a/2}{2\sin\pi a/2} = \cos\frac{\pi a}{2}$$
Hence
$$\frac{\Gamma(1+a/2)}{\Gamma(1+a)} = \cos\frac{\pi a}{2} \cdot \Gamma(1-a)\Big/\Gamma(1-a/2)$$
Similarly
$$\frac{\Gamma(1+a-b)\Gamma(b-a)}{\Gamma(1+a/2-b)\Gamma(b-a/2)} = \frac{\Gamma(1-(b-a))\Gamma(b-a)}{\Gamma(1-(b-\frac{a}{2}))\Gamma(b-\frac{a}{2})}$$
$$= \frac{\pi}{\sin\pi(b-a)} \Big/ \frac{\pi}{\sin\pi(b-a/2)}$$

$$= \frac{\sin \pi (b - a/2)}{\sin \pi (b-a)}$$

Now Dixon's Theorem

$${}_3F_2 \left[\begin{matrix} a, & b, & c; \\ 1+a-b, & 1+a-c, & \end{matrix} 1 \right] = \frac{\Gamma(1+a/2)\,\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+\frac{a}{2}-c)} \cdot \frac{\Gamma(1+a/2-b-c)}{\Gamma(1+a-b-c)}$$

$$= \frac{\cos \pi a/2 \cdot \Gamma(1-a)}{\Gamma(1-a/2)} \cdot \frac{\sin \pi (b - a/2)}{\sin \pi (b-a)} \cdot \frac{\Gamma(1+a-c)\,\Gamma(1+a/2-b-c)}{\Gamma(1+a/2-c)\,\Gamma(1+a-b-c)}$$

$$= \frac{\cos \pi a/2 \cdot \sin \pi (b-a/2)\,\Gamma(1-a)\,\Gamma(b-a/2)\,\Gamma(1+a-c)}{\sin \pi (b-a) \cdot \Gamma(1-a/2)\,\Gamma(b-a)\,\Gamma(1+a/2-c)\,\Gamma(1+a-b-c)}$$

Q. No. 4) Use the result in Ex. (3) to show that if n is a non negative integer

$${}_3F_2 \left[\begin{matrix} -2n, \alpha, 1-\beta-2n; \\ 1-\alpha-2n, \beta; \end{matrix} 1 \right] = \frac{(2n)!\,(\alpha)_n\,(\beta-\alpha)_n}{n!\,(\alpha)_{2n}\,(\beta)_n}$$

Solution: Put $a = -2n$, $b = \alpha$, $1-\beta-2n = c$

$\Rightarrow \beta = 1 - c - 2n$

in exercise (3) we get

$${}_3F_2 \left[\begin{matrix} -2n & \alpha, & 1-\beta-2n; \\ 1-\alpha-2n, & \beta; & \end{matrix} 1 \right]$$

$$= \frac{\cos(-\pi n) \sin \pi (\alpha + n)\,\Gamma(1+2n)\,\Gamma(\alpha+n)}{\sin \pi (\alpha+2n)\,\Gamma(1+n)\,\Gamma(\alpha+2n)} \cdot$$

$$\frac{\Gamma(1-2n-1+\beta+2n)}{\Gamma(1-n-1+\beta+2n)} \cdot \frac{\Gamma(1-n-\alpha-1+\beta+2n)}{\Gamma(1-2n-\alpha-1+\beta+2n)}$$

$$= \frac{(-1)^n\left[\sin(\pi\alpha)\cos(\pi n)+\sin\pi n\cdot\cos\pi\alpha\right]2n!\;\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\cdot\Gamma(\beta)\Gamma(\beta-\alpha+n)}{\left[\sin\pi\alpha\cdot\cos2\pi n+\sin2\pi n\cos\pi\alpha\right]\cdot n!\;\frac{\Gamma(\alpha+2n)}{\Gamma(\alpha)}\Gamma(\beta+n)\Gamma(\beta-\alpha)}$$

$$= \frac{(-1)^n(-1)^n\sin\pi\alpha\cdot 2n!\,(\alpha)_n\,\Gamma(\beta)\Gamma(\beta-\alpha+n)}{\sin\pi\alpha\cdot n!\,(\alpha)_{2n}\,\Gamma(\beta+n)\Gamma(\beta-\alpha)}$$

$$= \frac{(2n)!\,(\alpha)_n\,(\beta-\alpha)_n}{n!\,(\alpha)_{2n}\,(\beta)_n}$$ As Required.

Q.No. 5) with the aid of the formula in Ex.4 Prove Ramanujan's theorem

$${}_1F_1\left[\begin{matrix}\alpha;\\ \beta;\end{matrix}\; x\right]{}_1F_1\left[\begin{matrix}\alpha;\\ \beta;\end{matrix}\; -x\right] = {}_2F_3\left[\begin{matrix}\alpha,\ \beta-\alpha;\\ \beta,\ \tfrac{1}{2}\beta,\ \tfrac{1}{2}\beta+\tfrac{1}{2};\end{matrix}\; \frac{x^2}{4}\right]$$

Solution: $${}_1F_1(\alpha;\beta;x)\,{}_1F_1(\alpha;\beta;-x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_n(\alpha)_m}{(\beta)_n(\beta)_m}\frac{x^n}{n!}\frac{(-1)^m x^m}{m!}$$

Now using lemma.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m).$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_{n-m}(\alpha)_m}{(\beta)_{n-m}(\beta)_m}\frac{x^{n-m}(-1)^m}{(n-m)!}\frac{x^m}{m!}$$

$$\because\ (\alpha)_{n-m} = \frac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m} \qquad \left\{ (n-m)! = \frac{(-1)^m n!}{(-n)_m}\right.$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m}\cdot\frac{(1-\beta-n)_m}{(-1)^m(\beta)_m}\cdot\frac{(\alpha)_m}{(\beta)_m}\frac{(-n)_m}{n!(-1)^m}\frac{(-1)^m x^n}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_n(\alpha)_m}{(\beta)_n(\beta)_m}\frac{(1-\beta-n)_m}{(1-\alpha-n)_m}\cdot\frac{(-n)_m}{m!\,n!}x^n.$$

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{(-n)_m(\alpha)_m(1-\beta-n)_m}{(\beta)_m\ (1-\alpha-n)_m\ m!}\right)\cdot\frac{(\alpha)_n x^n}{n!(\beta)_n}$$

$$=\sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix}-n, & \alpha, & 1-\beta-n; \\ \beta, & 1-\alpha-n; & \end{matrix}\ 1\right]\frac{(\alpha)_n x^n}{n!(\beta)_n}$$

Replacing n by $2n$

$$=\sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix}-2n, & \alpha, & 1-\beta-2n; \\ \beta, & 1-\alpha-2n; & \end{matrix}\right]\frac{(\alpha)_{2n} x^{2n}}{(2n)!(\beta)_{2n}}$$

Now by using Exe. (4) we have.

$$=\sum_{n=0}^{\infty}\frac{(2n)!(\alpha)_n(\beta-\alpha)_n}{n!(\alpha)_{2n}(\beta)_n}\cdot\frac{(\alpha)_{2n}x^{2n}}{(2n)!(\beta)_{2n}}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta-\alpha)_n}{n!(\beta)_n 2^{2n}\left(\frac{\beta}{2}\right)_n\left(\frac{\beta+1}{2}\right)_n}\ x^{2n}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta-\alpha)_n}{(\beta)_n\left(\beta/2\right)_n\left(\beta/2+1/2\right)_n}\frac{\left(\frac{x^2}{2^2}\right)^n}{n!}$$

$$= {}_2F_3\left[\begin{matrix}\alpha, & \beta-\alpha; & \\ \beta, & 1/2\,\beta, & 1/2\,\beta+1/2; \end{matrix}\ \frac{x^2}{4}\right]$$

Q. No. 6) Let $r_n = {}_3F_2(-n, 1-a-n, 1-b-n; a, b; 1)$

use the result in Exe.(3) to show that

$r_{2n+1} = 0$ and.

$$r_{2n} = \frac{(-1)^n(2n)!\ (a+b-1)_{3n}}{n!(a)_n(b)_n(a+b-1)_{2n}}.$$

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Since exercise (3) is

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} ; 1\right] = \frac{\cos\frac{\pi a}{2} \cdot \sin\pi(b-a/2)}{\sin\pi(b-a)} \cdot \frac{\Gamma(1-a)\Gamma(b-a/2)}{\Gamma(1-\frac{1}{2}a)\Gamma(b-a)} \cdot \frac{\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a/2-c)\Gamma(1+a-b-c)} \quad (1) $$

Put

$a = -n$ and since $\cos\left(\frac{\pi n}{2}\right) = 0$

Hence $\gamma_{2n+1} = 0$ if n is odd

Now we have to evaluate γ_{2n}

$$ \therefore \gamma_{2n} = {}_3F_2(-2n, 1-a-2n, 1-b-2n, a, b ; 1) $$

So we put $a = -2n$, $b = 1-a-2n$, $c = 1-b-2n$ in (1)

$$ \gamma_{2n} = \frac{\cos(n\pi)\sin\pi.[1-a-2n+n]}{\sin\pi(1-a-2n+2n)} \cdot \frac{\Gamma(1+2n)\Gamma(1-a-n)}{\Gamma(1+n)\Gamma(1-a)} \cdot \frac{\Gamma(1-2n-1+b+2n) \cdot \Gamma(1-n-1+a+2n-1+b+2n)}{\Gamma(1-n-1+b+2n)\Gamma(1-2n-1+a+2n-1+b+2n)} $$

$$ = \frac{\cos n\pi \sin\pi(1-a-n)}{\sin\pi(1-a)} \frac{\Gamma(1+2n)\Gamma(1-a-n)}{\Gamma(1+n)\Gamma(1-a)} \cdot \frac{\Gamma(b)\Gamma(3n-1+a+b)}{\Gamma(b+n)\Gamma(2n+a+b-1)} $$

$$ \because \frac{\Gamma(1-a-n)}{\Gamma(1-a)} = \frac{(-1)^n}{(a)_n} $$

$$ = \frac{(-1)^n(-1)^n \sin\pi(1-a)}{\sin\pi(1-a)} \cdot \frac{2n!}{n!} \cdot \frac{(-1)^n (a+b-1)_{3n}}{(a)_n (b)_n (a+b-1)_{2n}} $$

$$ = \frac{(-1)^n \, 2n! \, (a+b-1)_{3n}}{n! \, (a)_n (b)_n \, (a+b-1)_{2n}} . $$

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Q. No. 7) with the aid of the result in exercise (6) show that

$${}_0F_2(-;a,b;t)\,{}_0F_2(-;a,b;-t)$$

$$= {}_3F_8\left[\begin{matrix} \tfrac12(a+b-1),\ \tfrac13(a+b),\ \tfrac13(a+b+1)\ ; & \\ a,b,\tfrac12 a,\ \tfrac12 a+\tfrac12,\ \tfrac12 b,\ \tfrac12 b+\tfrac12,\ \tfrac12(a+b-1),\ \tfrac12(a+b)\ ; & \dfrac{-27t^2}{64}\end{matrix}\right]$$

Solution: let $\psi = {}_0F_2(-;a,b:t)\,{}_0F_2(-;a,b;-t)$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^n(-1)^m t^m}{(a)_n(b)_n(a)_m(b)_m}\cdot\frac{1}{m!\,n!}$$

$$\because \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m).$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{t^{n-m}(-1)^m t^m}{(a)_{n-m}(b)_{n-m}(a)_m(b)_m}\cdot\frac{1}{m!\,(n-m)!}$$

$$\because \frac{1}{(n-m)!} = \frac{(-1)^m(-n)_m}{n!} \quad \& \quad (a)_{n-m} = \frac{(-1)^m(a)_n}{(1-a-n)_m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} t^{n-m}(-1)^m t^m(-n)_m \frac{(1-a-n)_m}{(-1)^m(a)_n}\cdot\frac{(1-b-n)_m}{(-1)^m(b)_n}\cdot\frac{1}{(a)_m(b)_m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{t^n(-1)^m(-n)_m(1-a-n)_m(1-b-n)_m}{m!\,n!\,(-1)^m\,(a)_n(b)_n(a)_m(b)_m}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{(-n)_m(1-a-n)_m(1-b-n)_m}{(a)_m(b)_m\,m!}\right)\frac{t^n}{(a)_n(b)_n\,n!}$$

by exercise (6)

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$$= \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{(a)_n (b)_n n!}$$

$$= \sum_{n=0} \frac{\gamma_{2n} \, t^{2n}}{(a)_{2n} (b)_{2n} (2n)!}$$ replace n by $2n$

using value of γ_{2n}

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! \, (a+b-1)_{3n}}{n! \, (a)_n (b)_n (a+b-1)_{2n}} \cdot \frac{t^{2n}}{(a)_{2n} (b)_{2n} (2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n} \cdot 3^{3n} \left(\frac{a+b-1}{3}\right)_n \left(\frac{a+b}{3}\right)_n \left(\frac{a+b+1}{3}\right)_n}{n! \, (a)_n (b)_n \, 2^{2n} \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n 2^{2n} \left(\frac{b}{2}\right)_n \left(\frac{b+1}{2}\right)_n} \cdot \frac{1}{2^{2n} \left(\frac{a+b-1}{2}\right)_n \left(\frac{a+b}{2}\right)_n}$$

$$= {}_3F_8 \left[\begin{matrix} 1/3(a+b-1), 1/3(a+b), 1/3(a+b+1) ; \\ a, b, \frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2}, \frac{a+b-1}{2}, \frac{a+b}{2} ; \end{matrix} \; \frac{-27t^2}{64} \right]$$