Derived Euler's Geometrical eq's

If $\bar{\omega}'$ is angular velocity after the three rotations.

$\boxed{K = K^x}$ $\boxed{I = I'}$ $\boxed{K' = K}$

$$\bar{\omega}' = \omega_\varphi K^x + \omega_\theta I^x + \omega_\psi k'$$

$$= \omega_\varphi(\sin\theta j^* + \cos\theta K) + \omega_\theta(\cos\psi i' - \sin\psi j') + \omega_\psi k'$$

$$= \omega_\varphi[\sin\theta(\sin\psi i' + \cos\psi j') + \cos\theta k'] + \omega_\theta \cos\psi i'$$
$$- \omega_\theta \sin\psi j' + \omega_\psi k'$$

$$= \omega_\varphi \sin\theta\sin\psi i' + \omega_\varphi \sin\theta\cos\psi j' + \omega_\varphi\cos\theta k' + \omega_\theta \cos\psi i'$$
$$- \omega_\theta \sin\psi j' + \omega_\psi k'$$

$$\bar{\omega}' = (\omega_\varphi \sin\theta\sin\psi + \omega_\theta\cos\psi) i' + (\omega_\varphi \sin\theta\cos\psi - \omega_\theta\sin\psi) j'$$
$$+ (\omega_\varphi\cos\theta + \omega_\psi) k'$$

Where,

$$\left\{\begin{array}{l}
\omega_x' = \omega_\varphi \sin\theta\sin\psi + \omega_\theta\cos\psi = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_y' = \omega_\varphi\cos\psi\sin\theta - \omega_\theta\sin\psi = \dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\
\omega_z' = \omega_\varphi\cos\theta + \omega_\psi = \dot{\varphi}\cos\theta + \dot{\psi}
\end{array}\right.$$

are known as Euler's geometrical eq's.

Euler's Dynamical eqs of motion

let a rigid body is rotating with angular velocity $\bar{\omega}$ abt a fixed point O, let OX, OY & OZ are principal axis, if L & $[I_{ij}]$ are angular momentum & inertia metrix then,

$$[L] = [I][\bar{\omega}]$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

∴ Abt principal axis, we have,

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = I_{zx} = I_{xz} = 0$$

(i.e product of inertia vanishes abt principal axis")

So,

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$L_x = I_{xx}\omega_x$ or $L_1 = I_1\omega_1$

$L_y = I_{yy}\omega_y$ or $L_2 = I_2\omega_2$

$L_z = I_{zz}\omega_z$ or $L_3 = I_3\omega_3$

So,

$$\bar{L} = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$$

$$\bar{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k} \rightarrow (1)$$

using Rotating axis theorem-

$$\left(\frac{d\bar{A}}{dt}\right)_f = \left(\frac{d\bar{A}}{dt}\right)_r + \bar{\omega} \times \bar{A}$$

So,

$$\left(\frac{d\bar{L}}{dt}\right)_f = \left(\frac{dL}{dt}\right)_r + \bar{\omega} \times \bar{L} \longrightarrow (2)$$

as, $\underline{L} = \underline{r} \times \underline{p} \longrightarrow$ linear mom.

$$\left(\frac{dL}{dt}\right)_f = \frac{d\bar{r}}{dt} \times \bar{P} + \bar{r} \times \frac{d\bar{P}}{dt}.$$

$$= \bar{r} \times p + \bar{r} \times m\bar{a}$$

$$= 0 + \bar{r} \times \bar{F}$$

$$= \bar{G} \text{ (say)} \longrightarrow (3)$$

$\because p = mv$

$\frac{d}{dt}(mv)$

$\therefore m\frac{dv}{dt}$

$m\bar{a}$

$$\left(\frac{d\bar{L}}{dt}\right)_r = \frac{d}{dt}(I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k})$$

$$= I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} \longrightarrow (4).$$

Now

$$\bar{\omega} \times \bar{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= \omega_2\omega_3(I_3 - I_2)\hat{i} + \omega_1\omega_3(I_1 - I_3)\hat{j} + \omega_1\omega_2(I_2 - I_1)\hat{k}$$

$$\longrightarrow (5)$$

using values from, (3), (4) & (5) into eq (2)

$$\bar{G} = I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} + \omega_2\omega_3(I_3 - I_2)\hat{i} + \omega_1\omega_3(I_1 - I_3)\hat{j} + \omega_1\omega_2(I_2 - I_1)\hat{k}$$

Now Comparing:-

$$\left.\begin{aligned} G_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3-I_2) \\ G_2 &= I_2\dot{\omega}_2 + \omega_1\omega_3(I_1-I_3) \\ G_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2-I_1) \end{aligned}\right] \rightarrow (6)$$

These are known as Euler dynamical eqns.

Deduction:-

for torque free motion $\Rightarrow \bar{G}=0$
then Euler's eqs becomes,

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3-I_2)=0 \rightarrow (7)$$
$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1-I_3)=0 \rightarrow (8)$$
$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2-I_1)=0 \rightarrow (9)$$

xply eq (7) by ω_1, (8) by ω_2, (9) by ω_3

$$I_1\omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3(I_3-I_2)=0$$
$$I_2\omega_2\dot{\omega}_2 + \omega_1\omega_2\omega_3(I_1-I_3)=0$$
$$I_3\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_2-I_1)=0$$

adding these, we get,

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0.$$

or $$\frac{1}{2}\frac{d}{dt}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]=0$$

$$\left(\frac{d}{dt}\frac{1}{2}\omega_1^2 = \frac{1}{2}2\omega_1\dot{\omega}_1\right.$$

Integ.

$$\frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] = \text{Const.}$$

$$\frac{1}{2}\left[I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}\right]\cdot\left[\omega_1\hat{i}+\omega_2\hat{j}+\omega_3\hat{k}\right]=c$$

$$\frac{1}{2}\,L\cdot\bar{\omega} = c \rightarrow (10)$$

This is known as energy theorem.

Q1:- Show that for a torque free motions magnitude of angular mom. is Constant,

Sol:-

for a torque-free motion, Euler's dynamical eq's are,

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = 0 \longrightarrow (1)$$

$$I_2 \dot{\omega}_2 - \omega_1 \omega_3 (I_3 - I_1) = 0 \longrightarrow (2)$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0 \longrightarrow (3)$$

xply (1) $I_1\omega_1$, (2) $I_2\omega_2$ & (3) $I_3\omega_3$

$$I_1^2 \omega_1 \dot{\omega}_1 - \omega_1 \omega_2 \omega_3 (I_1 I_2 - I_1 I_3) = 0$$

$$I_2^2 \omega_2 \dot{\omega}_2 - \omega_1 \omega_2 \omega_3 (I_2 I_3 - I_2 I_1) = 0$$

$$I_3^2 \omega_3 \dot{\omega}_3 - \omega_1 \omega_2 \omega_3 (I_3 I_1 - I_3 I_2) = 0$$

adding all these,

$$I_1^2 \omega_1 \dot{\omega}_1 + I_2^2 \omega_2 \dot{\omega}_2 + I_3^2 \omega_3 \dot{\omega}_3 = 0$$

$$\frac{1}{2} \frac{d}{dt} (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2) = 0$$

Integ.

$$\frac{1}{2} (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2) = c^x$$

$$(I_1\omega_1 \hat{i} + I_2\omega_2 \hat{j} + I_3\omega_3 \hat{k}) \cdot (I_1\omega_1 \hat{i} + I_2\omega_2 \hat{j} + I_3\omega_3 \hat{k}) = c$$

$$L \cdot L = c$$

$$|L^2| = c$$

$$|L| = c$$

$\Rightarrow$ Mag. of A.M is Const.

Q:2 In case of Spherical torque free motion, angular velocity is Constant.

Sol:-

∴ for a torque free motion Euler's dynamical eq's are,

$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0$

$I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1) = 0$

$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0$

for any Spherical Top.

$I_1 = I_2 = I_3 = I \neq 0$

$\Rightarrow I_1\dot{\omega}_1 = 0$; $I\dot{\omega}_2 = 0$, $I\dot{\omega}_3 = 0$

integ. all,

$\Rightarrow \omega_1 = c_1$

$\Rightarrow \omega_2 = c_2$

$\Rightarrow \omega_3 = c_3$

Therefore;

$\omega = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} = \text{Const.}$

$\Rightarrow$ Angular Vel is Const.

Q3:-

An ellipsoid torque free motion about its centre, set into motion at $t=0$ & $(n, 0, 3n)$ as Components of its angular velocity $\bar{\omega}$ along the Principal axis, $6A, 3A$ & A are M.I about the axis,

find angular velocity at any time t & show that when $t \to \infty$ the mag. of the angular velocity is $n\sqrt{5}$

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Sol:

for torque free motion, Euler's eq of motion are, $I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0 \rightarrow (1)$

$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \rightarrow (2)$

$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \rightarrow (3)$

using $I_1 = 6A$, $I_2 = 3A$, $I_3 = A$

We have,

$6A\dot{\omega}_1 - 2A\omega_2\omega_3 = 0$ } ÷ by 2

$3A\dot{\omega}_1 - A\omega_2\omega_3 = 0 \rightarrow (4)$

$3A\dot{\omega}_2 + 5A\omega_1\omega_3 = 0 \rightarrow (5)$

$A\dot{\omega}_3 - 3A\omega_1\omega_2 = 0 \rightarrow (6)$

(4) $5\omega_1 \Rightarrow$ $15A\omega_1\dot{\omega}_1 - 5A\omega_1\omega_2\omega_3 = 0$

(5) $\omega_2 \Rightarrow$ $3A\omega_2\dot{\omega}_2 + 5A\omega_1\omega_2\omega_3 = 0$

$15A\omega_1\dot{\omega}_1 + 3A\omega_2\dot{\omega}_2 = 0$

$3A(5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2) = 0$

$5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$

Integ

$\frac{5\omega_1^2}{2} + \frac{\omega_2^2}{2} = c_1'$

$5\omega_1^2 + \omega_2^2 = 2c_1'$ $\quad$ ($\because 2c_1' = c_1$

$5\omega_1^2 + \omega_2^2 = c_1 \rightarrow (7)$

at $t = 0$, $\omega_1 = n$, $\omega_2 = 0$

$5n^2 = c_1$

$5\omega_1^2 + \omega_2^2 = 5n^2$

$5\omega_1^2 = 5n^2 - \omega_2^2$

$\omega_1 = \sqrt{\frac{5n^2 - \omega_2^2}{5}} \rightarrow (8)$

from (4) & (6)

(4) $3\omega_1$, (6) ω_3.

$$9A\dot{\omega}_1\omega_1 - 3A\omega_1\omega_2\omega_3 = 0$$

$$-A\dot{\omega}_3\omega_3 \mp 3A\omega_1\omega_2\omega_3 = 0$$

$$9A\dot{\omega}_1\omega_1 - A\dot{\omega}_3\omega_3 = 0$$

$$9\dot{\omega}_1\omega_1 - \dot{\omega}_3\omega_3 = 0$$

integ.

$$9\frac{\omega_1^2}{2} - \frac{\omega_3^2}{2} = c_2'$$

$$9\omega_1^2 - \omega_3^2 = c_2.$$

at $t=0$, $\omega_1 = n$, $\omega_3 = 3n$.

$$9n^2 - 9n^2 = c_2$$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow 9\omega_1^2 = \omega_3^2$$

$$3\omega_1 = \omega_3$$

$$\omega_3 = 3\sqrt{\frac{5n^2 - \omega_2^2}{5}} \longrightarrow (10)$$

using these results, (8), (10) into (5)

$$3A\dot{\omega}_2 + 5A\omega_1\omega_3 = 0$$

$$3A\dot{\omega}_2 + 5A\left(\sqrt{\frac{5n^2-\omega_2^2}{5}}\right)\left(3\sqrt{\frac{5n^2-\omega_2^2}{5}}\right) = 0$$

$$3A\dot{\omega}_2 + 5A\left(3\left(\frac{5n^2-\omega_2^2}{5}\right)\right) = 0$$

$$3A\dot{\omega}_2 + 15A\left(\frac{5n^2-\omega_2^2}{5}\right) = 0$$

$$\frac{15A\dot{\omega}_2 + 15A(5n^2-\omega_2^2)}{5} = 0$$

$$15A\dot{\omega}_2 + 15A(5n^2-\omega_2^2) = 0$$

$$\dot{\omega}_2 + (5n^2 - \omega_2^2) = 0$$

$$\dot{\omega}_2 - \omega_2^2 + 5n^2 = 0$$

$$\frac{d\omega_2}{dt} = \omega_2^2 - 5n^2.$$

$$d\omega_2 = (\omega_2^2 - 5n^2)dt$$

$$\frac{d\omega_2}{\omega_2^2 - 5n^2} = dt$$

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$-\int \frac{d\omega_2}{5n^2 - \omega_2^2} = \int dt$$

$$-\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = t + c_1$$

$$= -\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = t + c_1$$

Put $t = 0, \omega_2 = 0$

$$c_1 = \frac{-1}{2\sqrt{5}n} \ln \frac{\sqrt{5}\cancel{n}}{\sqrt{5}\cancel{n}}.$$

$$= \frac{-1}{2\sqrt{5}n} \ln(1)^{\to 0}$$

$$c_1 = 0.$$

$$\therefore t = \frac{-1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2}$$

$$\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = -t$$

$$\ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = -2\sqrt{5}nt$$

$$\frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = e^{-2\sqrt{5}nt}$$

$$\frac{\cancel{\sqrt{5}n} + \omega_2 + \sqrt{5}n - \cancel{\omega_2}}{\cancel{\sqrt{5}n} + \omega_2 - \cancel{\sqrt{5}n} + \omega_2}$$

$$= \frac{e^{-2\sqrt{5}nt} + 1}{e^{-2\sqrt{5}nt} + 1}$$

$$\frac{\cancel{2}\sqrt{5}n}{\cancel{2}\omega_2} = \frac{e^{-2\sqrt{5}nt} + 1}{e^{-2\sqrt{5}nt} + 1}$$

$$\omega_2 = \sqrt{5}n \left[\frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1} \right]$$

also

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$-\int \frac{d\omega_2}{5n^2 - \omega_2^2} = \int dt$$

$$-\int \frac{d\omega_2}{(\sqrt{5}n)^2 - (\omega_2)^2} = \int dt$$

$$\frac{-1}{\sqrt{5}n} \tanh^{-1} \frac{\omega_2}{\sqrt{5}n} = t$$

$$\tanh^{-1} \frac{\omega_2}{\sqrt{5}n} = -\sqrt{5}nt$$

$$\frac{\omega_2}{\sqrt{5}n} = -\tanh \sqrt{5}nt$$

$$\omega_2 = -\sqrt{5}n \tanh \sqrt{5}nt$$

$$\omega_2 = n\sqrt{5}\left[\frac{e^{-\sqrt{5}nt}}{e^{-\sqrt{5}nt}}\left[\frac{e^{-\sqrt{5}nt} - e^{\sqrt{5}nt}}{e^{-\sqrt{5}nt} + e^{\sqrt{5}nt}}\right]\right]$$

$$\omega_2 = n\sqrt{5}\tanh\sqrt{5}nt$$

Now $\omega_1 = \sqrt{\dfrac{5n^2 - (-\sqrt{5}n\tanh\sqrt{5}nt)^2}{5}}$

$$= \sqrt{\frac{5n^2 - 5n^2\tanh^2\sqrt{5}nt}{5}}$$

$$= n\sqrt{1 - \tanh^2\sqrt{5}nt}$$

$$= n\sqrt{\operatorname{sech}^2\sqrt{5}nt}$$

$$\omega_1 = n\operatorname{sech}\sqrt{5}nt$$

$$\Rightarrow \omega_3 = 3n\operatorname{sech}\sqrt{5}nt$$

$$\bar{\omega} = \omega_1 i + \omega_2 j + \omega_3 k.$$

$$= n\operatorname{sech}\sqrt{5}nt\,\hat{i} - \sqrt{5}n\tanh\sqrt{5}nt\,j + 3n\operatorname{sech}\sqrt{5}nt\,k.$$

$t \longrightarrow \infty$

$\bar{\omega} = 0 - \sqrt{5}n(1) + 0.$ $\quad$ ($\tanh\infty = 1$, $\operatorname{sech}\infty = 0$)

$= -n\sqrt{5}$

$\bar{\omega} = \sqrt{5}n$. as Required.

Ans.

formulas,

$\star$ $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\frac{a+x}{a-x}.$

$\displaystyle\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\frac{x-a}{x+a}$

or $\checkmark$ $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}$ (using.