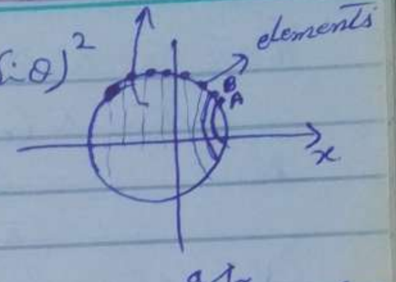
P-31

1 distances



$= \lim_{n \to \infty} \sum_{i=1}^{n} (\rho a^2 2\pi \sin\theta) d\theta \cdot (a \sin\theta)^2$

$= 2\pi a^4 \rho \int_0^{\pi} \sin^3\theta \, d\theta$

$= 2\pi a^4 \rho \int_0^{2\pi/2} \sin^3\theta \, d\theta$

$= 4\pi a^4 \rho \int_0^{\pi/2} \sin^3\theta \, d\theta$

$4\pi a^4 \rho \cdot \frac{2}{3} = 4\pi a^4 \cdot \frac{M}{4\pi a^2} \cdot \frac{2}{3}$

$= \cancel{2\pi a^4 \rho \, 2 \cdot 2/2}$ $= 2/3 \, Ma^2$

$= \frac{8a^4\pi}{3} \cdot \frac{M}{4\pi a^2}$

$= 2/3 \, Ma^2$

$V = 4\pi r^2$.

$x = a\sin\theta$

$\frac{x}{a} = \sin\theta$

$c = 2\pi r$

$c = 2\pi a \sin\theta$

$\rho = 2/3 \, Ma^2$

$V = 4/3 \, \pi r^3$

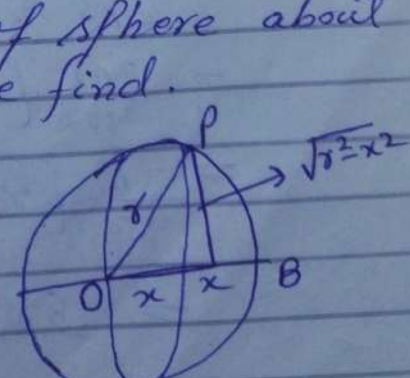
Q:- find M.I of solid sphere & hemisphere by a Technique other than techniques used in exp 7, 8, M.I of solid sphere about its diameter:-

Sol:- let M be the mass & ρ be its density & r be its radius, through which mass is uniformly distributed.

The solid sphere is assumed to be made up of large no's of discs whose radii vary from 0 to r.

let AB be diameter of sphere about which its M.I is to be find.

we first consider one such disc of thickness dx & mass dm then,



Then radius of disc $= \sqrt{r^2 - x^2}$.

$dA = \pi(\sqrt{r^2-x^2})^2 = \pi(r^2-x^2)$

Now, $dv = dx\,dy\,dz = dA\,dx$.

$= \pi(r^2-x^2)dx$.

by def. $\rho = \frac{m}{V} = \frac{dm}{dv} \Rightarrow dm = \rho\, dv$

$= \frac{m}{4/3\pi r^3} \cdot \pi(r^2-x^2)dx$. $\quad \left(\because \rho = \frac{m}{4/3\pi r^3}\right)$

if dI be M.I of disc, we have,

$dI = 1/2\ (\text{mass})(\text{radius})^2$.

$= 1/2\ dm(r^2-x^2)$

$= \frac{1}{2}\left[\frac{3m}{4r^3}(r^2-x^2)dx)(r^2-x^2)\right]$

$= \frac{3m}{8r^3}(r^2-x^2)^2 dx$.

M.I of solid sphere due to all disc is obtained by integrating above eq from 0 to r & ⊗ by ②.

if I be M.I of solid sphere, $I = 2\int_0^r dI$. $\quad$ ($\because$ disc make a sphere in this cases)

$= 2\int_0^r \frac{3m}{8r^3}(r^2-x^2)^2 dx$.

$= \frac{3m}{4r^3}\int_0^r (r^4 + x^4 - 2r^2x^2)dx$.

$= \frac{3m}{4r^3}\left[r^4 |x|_0^r + \left|\frac{x^5}{5}\right|_0^r - 2r^2\left|\frac{x^3}{3}\right|_0^r\right]$

$= \frac{3m}{4r^3}\left[r^5 + \frac{r^5}{5} - \frac{2}{3}r^5\right] = \frac{3m}{4r^3}\left(\frac{15r^5+3r^5-10r^5}{15}\right)$

$= \frac{8r^5}{15} \times \frac{3m}{4r^3} = \frac{2}{5}Mr^2$

Ans

exp3 M.I of rectangular lamina about an axis $\perp$ to the lamina & passing through G (centr of mass).

Sol:

Consider dimension of the rectangular lamina ABCD as $2a \times 2b$. let G is C.O.M & EG is $\perp$ axis.

Consider a rectangular mass element of length Δx & width Δy. also, r is $\perp$ to the axis of lamina EG

$r = \sqrt{x^2+y^2}$, $r^2 = x^2+y^2$.

Mass of element $= \rho \Delta x \Delta y$.

$$M.I = \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2)\, \rho\, dx\, dy.$$

$$= \rho \int_{-b}^{b} \left(\frac{x^3}{3} + y^2 x \right) \Big|_{-a}^{a} dy$$

$$= \rho \left[\int_{-b}^{b} \left[\frac{a^3}{3} + \frac{a^3}{3} + y^2 a + y^2 a \right] dy. \right.$$

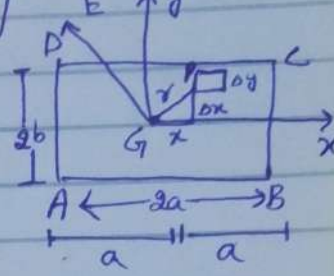
$$= \rho \left[\frac{2a^3}{3} \int_{-b}^{b} dy + 2a \int_{-b}^{b} y^2 dy \right)$$

$$= \rho\left[\frac{2a^3}{3}(b+b) + 2a\left(\frac{b^3}{3}+\frac{b^3}{3}\right)\right]$$

$$= \frac{m}{4ab}\left(\frac{4a^3b}{3} + \frac{4ab^3}{3}\right)$$

$$= \frac{m}{4ab}\cdot\frac{4ab(a^2+b^2)}{3}$$

$$= \frac{m(a^2+b^2)}{3}\text{. Ans}$$



$\left(\rho = \frac{m}{V} = \frac{m}{A} = \frac{m}{4ab}\right.$

$\frac{m}{3}(a^2+b^2)$

Converse of ⊥ axis theorem

i) if $I_{zz} = I_{xx} + I_{yy}$ then prove that the mass distribution, (rigid body is a plane lamina)

(2) prove that,

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

(i) Sol:-

Converse of ⊥ theorem:-

Consider, $I_{zz} = I_{xx} + I_{yy} \longrightarrow (1)$

we have to prove that the mass distribution is a plane lamina. Since,

$$I_{xx} = \Sigma m(y^2+z^2)$$
$$I_{yy} = \Sigma m(x^2+z^2)$$
$$I_{zz} = \Sigma m(x^2+y^2)$$

use in (1),

$$\Sigma m(x^2+y^2) = \Sigma m(y^2+z^2) + \Sigma m(x^2+z^2)$$
$$= \Sigma m(y^2+2z^2+x^2)$$
$$\Sigma mx^2 + \Sigma my^2 = \Sigma my^2 + 2\Sigma mz^2 + \Sigma mx^2$$
$$2\Sigma mz^2 = 0 \Rightarrow \Sigma mz^2 = 0$$

∀ distribution of mass for a single

Particle of mass m.
Since $m \neq 0$, $\Rightarrow z = 0$
$\Rightarrow$ It is a plane lamina. Proved

Principal Moment of inertia

The axis relative to which product of inertia vanishes (zero) are called principal axis. The M.I relative to the principal axis are called principal M.I

Another definations

if axis of rotation $(\underline{\omega})$ is $\parallel$ to angular momentum then the axis is known as principal axis. In this case; we write, $\underline{L} = n\underline{\omega}$, where n is const. also, we may write, $\underline{\omega} = \underline{\omega}\hat{a}$, $\underline{L} = \underline{L}\hat{a}$

Theorem:

Prove that in general there exist three principal axis through a pt of a rigid body.

(having same direction, angular mom. & ω)

Proof:

by defi.

$$\underline{L} = \sum_{i=1}^{n} \underline{r}' \times \underline{p} = \sum \underline{r} \times m\underline{v} = \sum m\underline{r} \times (\underline{\omega} \times \underline{r})$$

$$= \sum m(\underline{r}\cdot\underline{r}\,\underline{\omega} - \underline{r}\cdot\underline{\omega}\,\underline{r})$$

$$= \sum m(r^2\underline{\omega} - \underline{r}\cdot\underline{\omega}\,\underline{r})$$

$$n\underline{\omega} = \sum m r^2\underline{\omega} - \sum m\,\underline{r}\cdot\underline{\omega}\,\underline{r}$$

$$n\underline{\omega}\hat{a} = \sum m r^2\underline{\omega}\hat{a} - \sum m\underline{r}\cdot\underline{\omega}\hat{a}\,\underline{r}$$

$$\left(\sum m r^2 - n\right)\hat{a} = \sum m(\underline{r}\cdot\hat{a})\underline{r} \longrightarrow (1)$$

using $r^2 = x^2 + y^2 + z^2$

$$\hat{a} = \lambda\hat{i} + \mu\hat{j} + \nu\hat{k}$$

Where, λ, μ, ν are direction cosines of $\hat{a}$

$$\underline{r}\cdot\hat{a} = \lambda x + \mu y + \nu z$$

ang. mom & ang. vel $\parallel$ to (J) $\therefore$ $\Downarrow$ Pricipl axis

$\Rightarrow$ (1) becms

$$\left(\sum m r^2 - n\right)(\lambda\hat{i} + \mu\hat{j} + \nu\hat{k}) = \sum m(\lambda x + \mu y + \nu z)(x\hat{i} + y\hat{j} + z\hat{k})$$

Comparing ith Components

$$(\Sigma m r^2 - n)\lambda = \Sigma m(\lambda x + \mu y + \nu z)x$$

$$\left[\Sigma m(x^2+y^2+z^2) - n\right]\lambda = \Sigma m(\lambda x^2 + \mu xy + \nu zx)$$

$$\left[\Sigma m(y^2+z^2) - n\right]\lambda = \Sigma m \mu xy + \Sigma m \nu xz$$

$$\left[I_{xx} - n\right]\lambda - \Sigma m \mu xy - \Sigma m xz = 0.$$

$$(I_{xx} - n)\lambda + \mu I_{xy} + \nu I_{xz} = 0$$

$$(I_{xx} - n)\lambda + I_{xy}\mu + I_{xz}\nu = 0 \longrightarrow (2)$$

Similarly,

$$I_{yx}\lambda + (I_{yy} - n)\mu + I_{yz}\nu = 0 \longrightarrow (3)$$

$$I_{zx}\lambda + I_{zy}\mu + (I_{zz} - n)\nu = 0 \longrightarrow (4)$$

(A)

eq (2) → (4) are homogenious linear eq in λ, μ, ν, for their non-trivial sol.

λ, μ, ν کے 3 ریزلٹ expected ہیں یہاں سے $\hat{a}$ کی ویلیو نکلے گی جو کہ axis of روٹیشن ہے۔

$$\begin{vmatrix} I_{xx}-n & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-n & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-n \end{vmatrix} = 0$$

I_{xx}

it is cubic eq in "n" & is known as, characteristic eq of symtric inertia metrix, this mtrix eqtns has three real roots "n". say, n_1, n_2, n_3 corresponding to these three values of "n" we have from set A.

Three set of values of λ, μ, ν i.e three sets of d. cosines of $\hat{a}$. These three sets determine three principal

homogen eq give trivial sol (homog. & d.c are not simultaneously "0") [illegible] $\perp$ axs [illegible]

axis given by,

$$a_p = \lambda_p \hat{i} + \mu_p \hat{j} + \nu_p \hat{k}$$

$$p = 1, 2, 3$$

Theorem

Three principal axis through a pt of a rigid body are mutually orthogonal.

Proof

Take n_1, n_2, n_3 as roots of the characteristic eq.

$$\begin{vmatrix} I_{xx} - n & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - n & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - n \end{vmatrix} = 0$$

& let the three axis corresponding to n_1, n_2, n_3 are $\hat{a_1}, \hat{a_2}, \hat{a_3}$ from the eq.

$$(\Sigma m r^2 - n)\hat{a} = \Sigma m (\underline{r} \cdot \underline{a})\underline{r} \rightarrow (1)$$

$$(\Sigma m r^2 - n_1)\hat{a_1} = \Sigma m (\underline{r} \cdot \hat{a_1})\underline{r} \rightarrow (2)$$

$$(\Sigma m r^2 - n_2)\hat{a_2} = \Sigma m (\underline{r} \cdot \hat{a_2})\underline{r} \rightarrow (3)$$

$$(\Sigma m r^2 - n_3)\hat{a_3} = \Sigma m (\underline{r} \cdot \hat{a_3})\underline{r} \rightarrow (4)$$

Taking scalar product of eq (2) with $\hat{a_2}$ & eq (3) by $\hat{a_1}$ & Sub.

$$(\Sigma m r^2 - n_1)\hat{a_1} \cdot \hat{a_2} - (\Sigma m r^2 - n_2)\hat{a_2} \cdot a_1 = \Sigma (\underline{r} \cdot \hat{a_1})(\underline{r} \cdot \hat{a_2}) - \Sigma (r \cdot a_2)(r \cdot a_1)$$

$$\Rightarrow (\cancel{\Sigma m r^2} - n_1 - \cancel{\Sigma m r^2} + n_2)\hat{a_1} \cdot \hat{a_2} = 0$$

$$(n_2 - n_1)\hat{a_1} \cdot \hat{a_2} = 0$$

$$\Rightarrow \hat{a_1} \cdot \hat{a_2} = 0 \qquad \because n_2 - n_1 \neq 0, \quad n_2 \neq n_1$$

Similarly

$$\hat{a_2} \cdot \hat{a_3} = 0$$

$$\& \quad \hat{a_3} \cdot \hat{a_1} = 0$$

"Three axis are mutually $\perp$" // Hence proved

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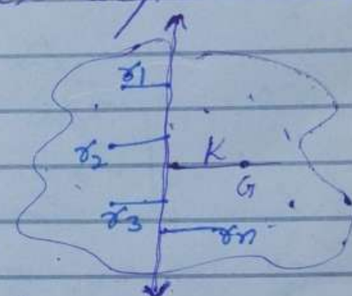
Radius of Gyration

radius in terms of particular shape

⇒ Radius of Gyration of a rigid body about a given axis is ~~called~~ ⊥ distance of the pt P from the axis where ~~its~~ total mass of the body were concentrated, the body shall have the same M.I as it has with the actual distri-bution of mass.

The radius of gyration is represented by K.



$$I' = I$$

$$MK^2 = \Sigma m d_i^2$$

$$= \Sigma m r_i^2$$

$$= m r_1^2 + m r_2^2 + \cdots m r_n^2$$

$$= m(r_1^2 + r_2^2 + r_3^2 + \cdots r_n^2) \quad (1)$$

$\therefore M = m + m + m \cdots m$ (n times)

$M = mn$

(1) becomes,

$$\cancel{m} n K^2 = \cancel{m}(r_1^2 + r_2^2 + \cdots r_n^2)$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots r_n^2}{n}} \longrightarrow (2)$$

Where n is the no. of particles each of mass m (equal masses in this case) & $r_1, r_2 \cdots r_n$ be their ⊥ distances from the axis of rotation.

from (2)

in 2D

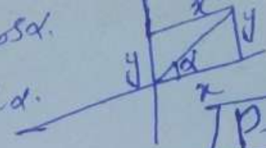
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos^2\alpha + \cos^2\beta = 1$

Basically $\cos$, $\sin$

$\frac{x}{R} = \cos\alpha$

$\frac{y}{R} = \sin\alpha$



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Radius of gyration of a body about an axis is equal to the root mean square distance of the constituent particles of the rigid body.

Momentum Ellipsoid :-

ellipse کو جب کسی ایکسز کے گرد rotate کریں تو وہ ellipsoid بن جاتی ہے یعنی 3D میں

2D → 3D

We know that M.I of a rigid body about a line with direction cosines λ, μ, ν is

$$I_{xx}\lambda^2 + I_{yy}\mu^2 + I_{zz}\nu^2 + 2\lambda\mu I_{xy} +$$

$$2\lambda\nu I_{xz} + 2\mu\nu I_{yz} = I_l \longrightarrow (1)$$

$I_{xx} = A$, $I_{yy} = B$, $I_{zz} = C$, $I_{yz} = D$, $I_{xz} = E$, $I_{xy} = F$

if $P(x, y, z)$ is any pt on this line &

$|OP| = r$

$\lambda = \frac{x}{r}$, $\mu = \frac{y}{r}$, $\nu = \frac{z}{r}$ use in (1)

$$I_{xx}\frac{x^2}{r^2} + I_{yy}\frac{y^2}{r^2} + I_{zz}\frac{z^2}{r^2} + \frac{2xy}{r^2}I_{xy}$$

$$+ 2\frac{xz}{r^2}I_{xz} + \frac{2yz}{r^2}I_{yz} = I_l.$$

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy$$

$$+ 2I_{xz}xz + 2I_{yz}yz = Ir^2 \longrightarrow (2)$$

If Ir^2 is taken as constant, then eq (2) represents an