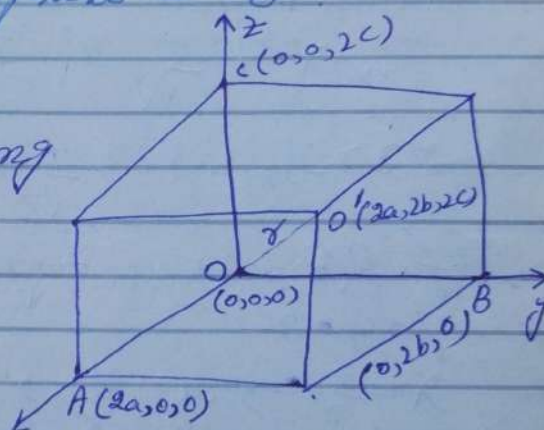
ellipsoid known as momentum ellipsoid or ellipsoid of inertia.

Ans

Q:- A uniform solid rectangular rigid body/block is of mass M & dimension $2a \times 2b \times 2c$. find eq of its momental ellipsoid for a corner 'O' of the block, (by taking 'O' as origin & coordinate axes along the edges of the block.) & hence, determine M.I about OO', O' is the pt diagonally opposite to 'O'.

Sol:-



Let $2a, 2b, 2c$ be along x-axis, y-axis, & z-axis

using,

$$I_{xx} = \int_V \rho d^2 dv.$$

$$= \int_0^{2c}\int_0^{2b}\int_0^{2a} \rho(y^2+z^2)\,dx\,dy\,dz.$$

★ $V = 2a \times 2b \times 2c = 8abc$ of cube

$$= \rho\int_0^{2c}\int_0^{2b}\left(y^2x+z^2x\Big|_0^{2a}\right)dy\,dz$$

$$= \rho\int_0^{2c}\int_0^{2b}(y^2 2a+z^2 2a)\,dy\,dz$$

$$= \rho\int_0^{2c}\left|2a\frac{y^3}{3}+2az^2y\right|_0^{2b}dz$$

$$= \rho \int_0^{2c} \left(2a \frac{8b^3}{3} + 2a z^2 \cdot 2b \right) dz$$

$$= \rho \left[\frac{16ab^3}{3} \left| z \right|_0^{2c} + 4ab \left| \frac{z^3}{3} \right|_0^{2c} \right]$$

$$= \rho \left[\frac{16ab^3}{3}(2c) + 4ab\left(\frac{8c^3}{3}\right) \right]$$

$$= \frac{M}{8abc} \cdot 32\, abc \left(\frac{b^2+c^2}{3} \right)$$

$$= \frac{4}{3} M (b^2+c^2)$$

Similarly,

$$I_{yy} = \frac{4}{3} M(a^2+c^2) \;\&\; I_{zz} = \frac{4}{3} M(a^2+b^2)$$

$$I_{yz} = -\int \rho\, yz\, dv$$

$$= -\rho \int_0^{2c}\int_0^{2b}\int_0^{2a} yz\, dx\,dy\,dz$$

$$= -\rho \left| \frac{z^2}{2} \right|_0^{2c} \left| \frac{y^2}{2} \right|_0^{2b} \left| x \right|_0^{2a}$$

$$= -\rho \left[\frac{4c^2}{2} \right] \left[\frac{4b^2}{2} \right] \left[2a \right]$$

$$= -8ab^2c^2 \frac{M}{8abc}$$

$$= -Mbc.$$

Similarly,

$$I_{zx} = -Mca \;\&\; I_{xy} = -Mab.$$

using the eq of momemental ellipsoid;

$$I_{xx} x^2 + I_{yy} y^2 + I_{zz} z^2 + I_{xy}(2xy) + 2yz I_{yz}$$

$$+ 2zx I_{zx} = I r^2 \longrightarrow (1)$$

$$\Rightarrow \left[\frac{4}{3}M(b^2+c^2)x^2+\frac{4}{3}M(a^2+c^2)y^2+\frac{4}{3}M(a^2+b^2)z^2\right]$$
$$-2M[abxy+bcyz+acxz]=Ir^2$$

Which is required eq of moment ellipsoid.

ii) To find M.I about a line O', we have $O'(2a, 2b, 2c)$ using $x=2a$, $y=2b$, $z=2c$ & $r^2=|OO'|^2=4(a^2+b^2+c^2)$

$$\frac{4}{3}M[(b^2+c^2)4a^2+(a^2+c^2)4b^2+(a^2+b^2)4c^2]$$
$$-2M[ab.4ab+bc.4bc+ac.4ac]$$
$$=4(a^2+b^2+c^2)I_{OO'}$$

$$4/3\, M(4a^2b^2+4a^2c^2+4a^2b^2+4b^2c^2+4a^2c^2+4b^2c^2)$$
$$-8M(a^2b^2+b^2c^2+c^2a^2)=I_{OO'}(4(a^2+b^2+c^2))$$
$$4(a^2+b^2+c^2)I_{OO'}=\frac{32m}{3}[a^2b^2+b^2c^2+c^2a^2]$$
$$-8m(a^2b^2+b^2c^2+c^2a^2)$$
$$=\frac{24m}{3}\frac{(a^2b^2+b^2c^2+c^2a^2)}{4(a^2+b^2+c^2)}$$
$$=6/3\;\frac{2m(a^2b^2+b^2c^2+c^2a^2)}{a^2+b^2+c^2}$$

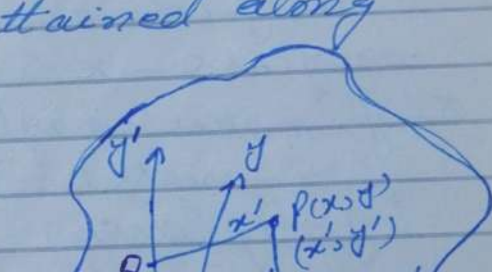
Theorems

i) Show that for 2·D mass distribution (lamina), one of the principal axis at O is inclined at an angle θ to the x-axis through O. S.t.

$$\tan 2\theta = \frac{2F}{B-A}$$

$A = I_{xx}$, $B = I_{yy}$, $F = I_{xy}$.

ii) Show that max & min values of M.I at O are attained along principal axis.



Proof:- Let us consider an arbitrary particle of mass m at a pt P whose coordinate w.r.t axes are (x, y). Then

M.I about x-axis,

$$\left.\begin{aligned} &= I_{xx} = A = \Sigma m y^2 \\ &\text{,, ,, } y\text{-axis} = B = \Sigma m x^2 \\ &\text{Product of Inertia.} \\ &I_{xy} = -\Sigma m x y = F \end{aligned}\right\} \rightarrow (1)$$

نوٹ:

$I_{xx} = \Sigma m y^2$

$I_{yy} = \Sigma m x^2$

$I_{x'x'} = \Sigma m y'^2$

$I_{y'y'} = \Sigma m x'^2$

$I_{xy} = -\Sigma m x y$

$I_{x'y'} = -\Sigma m x' y'$

We take another set of $\perp$ axis, Ox', Oy' s.t. Ox' is inclined at angle θ. With the coordinate x-axis, If (x', y') are coordinates of particle P w.r.t $x'y'$ system. Then,

$$A^* = I_{x'x'} = \Sigma m y'^2$$

$$B^* = I_{y'y'} = \Sigma m x'^2$$

$$F^* = I_{x'y'} = -\Sigma m x'y'$$

eq of line

$$\frac{|ax_1 + by_1 + c|}{\sqrt{x^2+y^2}}$$

$\perp$ distance b/w a pt & a line

slope $= \tan\theta$

$y - y_1 = m(x - x_1)$

Now, eq of line ox' is $y - 0 = \tan\theta(x - 0)$

$\Rightarrow y\cos\theta - x\sin\theta = 0 \longrightarrow$ (2)

($\therefore y$ makes angle $\theta + \pi/2$ with x-axis)

by replacing θ by $\pi/2 + \theta$ in (2)

$$y\cos(\pi/2 + \theta) - x\sin(\pi/2 + \theta) = 0$$

$$-y\sin\theta - x\cos\theta = 0$$

$$\Rightarrow x\cos\theta + y\sin\theta = 0 \longrightarrow (3)$$

$\therefore y' = |PL|$ is $\perp$ distance of P from line ox'

$$y' = \frac{|y\cos\theta - x\sin\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |y\cos\theta - x\sin\theta|$$

Similarly

$$x' = |PQ|$$

$$\frac{|y\sin\theta + x\cos\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |x\cos\theta + y\sin\theta|$$

Now $I_{x'x'} = \Sigma m y'^2 = \Sigma m (y\cos\theta - x\sin\theta)^2$

$= \Sigma m(y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta$

$= \cos^2\theta\, \Sigma m y^2 + \sin^2\theta\, \Sigma m x^2 + \sin 2\theta$
$(-\Sigma m xy)$

$= \cos^2\theta\, I_{xx} + \sin^2\theta\, I_{yy} + \sin 2\theta\, I_{xy}$.

$= \cos^2\theta (A) + \sin^2\theta (B) + \sin 2\theta (F)$

$= A\cos^2\theta + B\sin^2\theta + F\sin 2\theta$.

$I_{y'y'} = \Sigma m x'^2 = \Sigma m (y\sin\theta + x\cos\theta)^2$.

$= \Sigma m(x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta)$

$= \cos^2\theta\, \Sigma m x^2 + \sin^2\theta\, \Sigma m y^2 - \sin 2\theta(-\Sigma m xy)$

$= \cos^2\theta\, I_{yy} + \sin^2\theta\, I_{xx} - \sin 2\theta\, I_{xy}$.

$= B\cos^2\theta + A\sin^2\theta - \sin 2\theta F$.

$= A\left(\frac{1-\cos 2\theta}{2}\right) + B\left(\frac{1+\cos 2\theta}{2}\right) - \sin 2\theta F$

$= \frac{1}{2}(A+B) + \frac{1}{2}(B-A)\cos 2\theta - F\sin 2\theta$.

$I_{x'y'} = -\Sigma m x'y' = -\Sigma m (x\cos\theta + y\sin\theta)$
$(y\cos\theta - x\sin\theta)$

× & ÷ by '2' to comp. formula

$= -\Sigma m(xy\cos^2\theta - x^2\sin\theta\cos\theta + y^2\sin\theta\cos\theta$.
$- xy\sin^2\theta)$

$= -\Sigma m\left[xy(\underbrace{\cos^2\theta - \sin^2\theta}_{=\cos 2\theta}) - x^2\frac{\sin 2\theta}{2} + y^2\frac{\sin 2\theta}{2}\right]$

$= -\frac{1}{2}\sin 2\theta \sum my^2 + \frac{1}{2}\sin 2\theta \sum mx^2 + \cos 2\theta(-\sum mxy$

$= -\frac{1}{2}\sin 2\theta A + \frac{1}{2}\sin 2\theta B + \cos 2\theta F$

$I_{x'y'} = \left(\frac{B-A}{2}\right)\sin 2\theta + F\cos 2\theta.$

$\therefore$ ox', oy' are taken as principal axis,

$\therefore I_{x'y'} = 0 \rightarrow$ by defi of principl axis. product of inertia $\rightarrow$ zero

$\frac{B-A}{2}\sin 2\theta + F\cos 2\theta = 0$

$\Rightarrow \tan 2\theta = \frac{-2F}{B-A} = \left|\frac{2F}{B-A}\right|$

$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2F}{B-A}\right)$

Now

$I_{x'x'} = A\cos^2\theta + B\sin^2\theta + F\sin 2\theta.$

$= \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos 2\theta + F\sin 2\theta. \rightarrow (A)$

&

$I_{y'y'} = \frac{1}{2}(A+B) + \frac{1}{2}(B-A)\cos 2\theta - F\sin 2\theta \rightarrow (B)$

for extreme values of $I_{x'x'}$ & $I_{y'y'}$

$\frac{d}{d\theta}(I_{x'x'}) = 0$, $\frac{d}{d\theta}(I_{y'y'}) = 0$

$\because$

$= A\left(\frac{1+\cos 2\theta}{2}\right) + B\left(\frac{1-\cos 2\theta}{2}\right)$

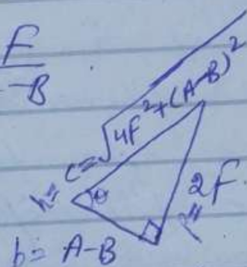
$= \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos 2\theta.$

$$\frac{d}{d\theta}(I_{x'x'}) = \frac{d}{d\theta}\left[\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\cos 2\theta + F\sin 2\theta\right] = 0$$

$$\Rightarrow -(A-B)\sin 2\theta + 2F\cos 2\theta = 0$$

$$\Rightarrow (A-B)\tan 2\theta = 2F.$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2F}{A-B}.$$



$$\Rightarrow \sin 2\theta = \frac{p}{h} = \frac{2F}{\sqrt{4F^2+(A-B)^2}}$$

$$\Rightarrow \cos 2\theta = \frac{b}{h} = \frac{A-B}{\sqrt{4F^2+(A-B)^2}} \quad \text{put in } (A)$$

$$\tan 2\theta = \frac{2F}{B-A}$$

$$I_{x'x'} = \tfrac{1}{2}(A+B) + \tfrac{1}{2}\frac{(A-B)^2}{\sqrt{4F^2+(A-B)^2}} + \frac{2F^2}{\sqrt{4F^2+(A-B)^2}}$$

$$= \tfrac{1}{2}(A+B) + \frac{(A-B)^2 + 4F^2}{2\sqrt{4F^2+(A-B)^2}}$$

$$= \frac{A+B}{2} \pm \frac{1}{2}\sqrt{4F^2+(A-B)^2}$$

$I_{y'y'}$ یوں یہاں (−) آجاتا ہے

As, these are the relation for the inclination of principal axis about coordinate axis so, the extreme values of $I_{x'x'}$, $I_{y'y'}$ are attained in the direction of principal axes.

Now

$$\frac{d}{d\theta}(I_{y'y'}) = -\tfrac{1}{2}(B-A)\,2\sin 2\theta - 2F\cos 2\theta = 0$$

$$\Rightarrow \tan 2\theta = \frac{-2F}{A-B}.$$

A

Theorem: (Result)

find M.I & Product of inertia for principal axis (Prove that product of inertia of principal axis are zero.

Proof:-

let $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are principal axis. as principal axis are mutually $\perp$ so, we can take coordinate axis along the principal axis then, $\underline{r} = \underline{op} = x\hat{a}_1 + y\hat{a}_2 + z\hat{a}_3$

$$r^2 = x^2 + y^2 + z^2$$

from Relation, $(\Sigma mr^2 - n)\hat{a} = \Sigma m(\underline{r}\cdot\hat{a})\underline{r}$

we have,

$$(\Sigma mr^2 - n_1)\hat{a}_1 = \Sigma m(\underline{r}\cdot\hat{a}_1)\underline{r} \longrightarrow (1)$$

$$(\Sigma mr^2 - n_2)\hat{a}_2 = \Sigma m(\underline{r}\cdot\hat{a}_2)\underline{r} \longrightarrow (2)$$

$$(\Sigma mr^2 - n_3)\hat{a}_3 = \Sigma m(\underline{r}\cdot\hat{a}_3)\underline{r} \longrightarrow (3)$$

by (1)

$$(\Sigma m(x^2+y^2+z^2) - n_1)\hat{a}_1 = \Sigma mx(xa_1 + ya_2 + za_3)$$

$$\Rightarrow (\Sigma m(y^2+z^2) - n_1)\hat{a}_1 - \Sigma mxy\hat{a}_2 - \Sigma mxz\hat{a}_3 = 0$$

Comparing,

$$\Sigma m(y^2+z^2) - n_1 = 0$$

$$-\Sigma mxy = 0 \; ; \quad -\Sigma mxz = 0$$

$$\Rightarrow n_1 = \Sigma m(y^2+z^2)$$

$$\Rightarrow I_{xx} = n_1 = \Sigma m(y^2+z^2)$$

&

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = I_{zx} = I_{xz} = 0$$

&

$$I_{yy} = \Sigma m(x^2+z^2) = n_2$$

$$I_{zz} = \Sigma m(x^2+y^2) = n_3$$

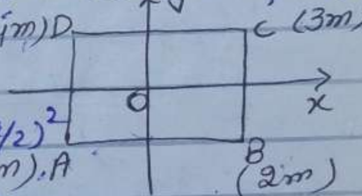
Q:

A square of side 'a' has particles of masses $m, 2m, 3m$ & $4m$ at its vertices, show that principal M.I at the centre of the square are $2ma^2$, $3ma^2$ & $5ma^2$. Also, find the direction of the principal axis,

Sol:

Consider a square ABCD of side 'a' with centre at origin. Consider coordinate axis as shown. let the masses $m, 2m, 3m$ & $4m$ be attached at the vertices:

$A(-a/2, -a/2)$, $B(a/2, -a/2)$, $C(a/2, a/2)$, $D(-a/2, a/2)$



$$A = \sum_{i=1}^{4} m_i y_i^2 = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2$$

$$= m(-a/2)^2 + 2m(-a/2)^2 + 3m(a/2)^2 + 4m(a/2)^2$$

$$= \frac{a^2}{4}[m + 2m + 3m + 4m] = \frac{10a^2 m}{4} = \frac{5ma^2}{2}.$$

$\tan 2\theta = \frac{2F}{A-B}$ P. axs

$I_{Ox'} = I_{x'x'} = A^*$
$= A\cos^2\theta + B\sin^2\theta + F\sin 2\theta$

Now,

$$B = \sum_{i=1}^{4} m_i x_i^2 = m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2 + m_4 x_4^2$$

$$= m(-a/2)^2 + 2m(a/2)^2 + 3m(a/2)^2 + 4m(-a/2)^2$$

$$= \frac{5ma^2}{2}$$

$$F = -\sum_{i=1}^{4} m_i x_i y_i$$

$$= -[m_1 x_1 y_1 + m_2 x_2 y_2 + m_3 x_3 y_3 + m_4 y_4 x_4]$$

$$= -\left[m\frac{a^2}{4} + 2m\left(-\frac{a^2}{4}\right) + 3m\left(\frac{a^2}{4}\right) + 4m\left(-\frac{a^2}{4}\right)\right]$$

$$= -\left[\frac{ma^2}{4} - \frac{2ma^2}{4} + \frac{3ma^2}{4} - \frac{4ma^2}{4}\right]$$

$$= \frac{ma^2}{2}$$

$$\tan 2\theta = \frac{2F}{A-B} = \infty = 2\theta = \pi/2 = \theta = \pi/4$$