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Saalschütz Theorem:

If n is a non-negative integer and if a, b, c are independent of n, then

$$ {}_3F_2\left[\begin{matrix}-n, a, b;\\ c, (1-c+a+b-n);\end{matrix} 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} $$

Proof: Since

$$ F[c-a, c-b; c; z) = (1-z)^{a+b-c} F(a, b; c; z) $$

$$ \Rightarrow \sum_{n=0}^{\infty} \frac{(c-a)_n (c-b)_n}{(c)_n} \frac{z^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(c-a-b)_n z^n}{n!}\right]\left[\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}\right] $$

$$ = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m (b)_m (c-a-b)_n}{(c)_m\, m!\, n!} z^{n+m} $$

Now by using Lemma

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m, m-n) $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_m (b)_m (c-a-b)_{n-m}}{(c)_m\, m!\, (n-m)!} z^n \quad \text{——(1)} $$

Since $(\alpha)_{n-m} = \dfrac{(-1)^m (\alpha)_n}{(1-\alpha-n)_m}$

Put $\alpha = 1$

$$ (1)_{n-m} = \frac{(-1)^m (1)_n}{(-n)_m} \Rightarrow (n-m)! = \frac{(-1)^m n!}{(-n)_m} $$

$$ \frac{1}{(n-m)!} = \frac{(-n)_m}{(-1)^m n!} $$

$\alpha = c-a-b$

$$ (c-a-b)_{n-m} = \frac{(-1)^m (c-a-b)_n}{(1-c+a+b-n)_m} $$

using these values in (1) we have.

$$ = \sum_{n=0}^{\infty}\sum_{n=0}^{n} \frac{(a)_m (b)_m}{(c)_m\, m!} \frac{(-n)_m}{n!} \frac{(c-a-b)_n}{(1-c+a+b-n)_m} z^n $$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \frac{(-n)_m (a)_m (b)_m}{(c)_m (1-c+a+b-n)_m} \cdot \frac{1}{m!} \right] \frac{(c-a-b)_n}{n!} z^n$$

$$= \sum_{n=0}^{\infty} {}_3F_2 \left[\begin{matrix} -n, a, b; \\ c, 1-c+a+b-n; \end{matrix} 1 \right] \frac{(c-a-b)_n}{n!} z^n$$

Comparing the coefficients of $\frac{z^n}{n!}$

$$\frac{(c-a)_n (c-b)_n}{(c)_n} = {}_3F_2 \left[\begin{matrix} -n, a, b; \\ c, 1-c+a+b-n; \end{matrix} 1 \right] (c-a-b)_n$$

$$* \sum_{m=0}^{n} \frac{(-n)_m (a)_m (b)_m}{(c)_m (1-c+a+b-n)_m} \frac{1}{m!}$$

Since Remaing terms are zero so we take $\sum_{m=0}^{n}$ instead of $\sum_{m=0}^{\infty}$

$$\Rightarrow {}_3F_2 \left[\begin{matrix} -n, a, b; \\ c, 1-c+a+b-n; \end{matrix} 1 \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}$$

Whipple's Theorem: If n is a non-negative integer and if b & c are independent of n, then prove that

$$ {}_3F_2\left[\begin{matrix} -n, b, c; \\ 1-b-n, 1-c-n; \end{matrix} x\right] = (1-x)^n {}_3F_2\left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, 1-b-c-n; \\ 1-b-n, 1-c-n; \end{matrix} \frac{-4x}{(1-x)^2}\right] $$

Proof: Consider a hypergeometric function

$$ {}_2F_1\left[\begin{matrix} b, c; \\ b+c; \end{matrix} t(1-x+xt)\right] $$

$$ \because (a-b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r $$

$$ = \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(b+c)_n} \cdot \frac{[(1-x)+xt]^n t^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(b)_n (c)_n}{(b+c)_n} \cdot \frac{n!}{(n-m)!\, m!} \frac{(1-x)^{n-m}(xt)^m t^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(b)_n (c)_n}{(b+c)_n} \frac{(1-x)^{n-m} \cdot x^m t^{n+m}}{(n-m)!\, m!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} \frac{(b)_{n-m} (c)_{n-m}}{(b+c)_{n-m}} \frac{(1-x)^{n-2m} x^m t^n}{(n-2m)!\, m!} $$

by Lemma

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} A(m, n-m). $$

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m, n-m) $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} (m, n-2m) $$

$$ \cdots\cdots \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} (m, n-2m) $$

Now using formula

$$ (\alpha)_{n-m} = \frac{(-1)^m (\alpha)_n}{(1-\alpha-n)_m} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} \frac{(-1)^m (b)_n (-1)^m (c)_n}{(1-b-n)_m (1-c-n)_m} \cdot \frac{(1-b-c-n)}{(-1)^m (b+c)_n} \cdot \frac{(1-x)^{n-2m} x^m t^m}{m!} $$

Since $(n-m)! =$

$$(n-2m)! = \frac{(-1)^{2m} n!}{(-n)_{2m}}$$

So

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(-1)^m (b)_n}{(1-b-n)_m} \frac{(-1)^m (c)_n}{(1-c-n)_m} \frac{(1-b-c-n)_m}{(-1)^m (b+c)_n} \frac{(-n)_{2m} (1-x)^{n-2m} x^m t^n}{m!\, n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(-1)^m (b)_n (c)_n (1-b-c-n)_m \cdot 2^{2m} \left(\frac{-n}{2}\right)_m \left(\frac{-n+1}{2}\right)_m}{(1-b-n)_m (1-c-n)_m (b+c)_n}$$

$$\cdot \frac{(1-x)^n (1-x)^{-2m} x^m t^n}{m!\, n!}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{[n/2]} \frac{(-\tfrac{1}{2}n)_m (-\tfrac{1}{2}n+\tfrac{1}{2})_m (1-b-c-n)_m}{(1-b-n)_m (1-c-n)_m\, m!} \left(\frac{-4x}{(1-x)^2}\right)^m \right]$$

$$\cdot \frac{(b)_n (c)_n}{(b+c)_n} \cdot \frac{(1-x)^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} (1-x)^n {}_3F_2 \left[\begin{matrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, 1-b-c-n; \\ 1-b-n, 1-c-n; \end{matrix} \; \frac{-4x}{(1-x)^2} \right]$$

$$\cdot \frac{(b)_n (c)_n}{(b+c)_n} \frac{t^n}{n!} \quad \text{--- (1)}$$

Again consider a hypergeometric function

$${}_2F_1 \left[\begin{matrix} b, c; \\ b+c; \end{matrix} \; t(1-x+xt) \right]$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(b+c)_n} t^n \frac{(1-x+xt)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(b)_n (c)_n}{(b+c)_n} \cdot \frac{n!}{(n-m)!\, m!} (1)^{n-m} [-x(1-t)]^m \cdot \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(b)_n (c)_n}{(b+c)_n} \frac{(-1)^m x^m (1-t)^m t^n}{(n-m)!\, m!}$$

using
$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m, n+m)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b)_{n+m} (c)_{n+m}}{(b+c)_{n+m}} \frac{(-1)^m x^m (1-t)^m t^{n+m}}{n!\, m!}$$

using $(\alpha)_{m+n} = (\alpha+m)_n (\alpha)_m$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b+m)_n (b)_m (c+m)_n (c)_m}{(b+c+m)_n (b+c)_m} \frac{(-1)^m x^m (1-t)^m t^{n+m}}{n!\, m!}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(b+m)_n (c+m)_n}{(b+c+m)_n} \frac{t^n}{n!} \right) \frac{(b)_m (c)_m}{(b+c)_m} \frac{(-1)^m x^m (1-t)^m t^m}{m!}$$

$$= \sum_{m=0}^{\infty} {}_2F_1 \left[\begin{matrix} b+m, c+m; \\ b+c+m; \end{matrix}\ t \right] \frac{(-1)^m (b)_m (c)_m [xt(1-t)]^m}{m!}$$

since.

$$F(a,b,c,z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

$$= \sum_{m=0}^{\infty} (1-t)^{b+c+m-b-m-c-m} {}_2F_1 \left[\begin{matrix} b+c+m-b-m, b+c+m-c-m \\ b+c+m \end{matrix}; t \right] \cdot \frac{(-1)^m (b)_m (c)_m (xt(1-t))^m}{(b+c)_m\, m!}$$

$$= \sum_{m=0}^{\infty} (1-t)^{-m} {}_2F_1 \left[\begin{matrix} c, b; \\ b+c+m \end{matrix}\ t \right] (-1)^m (b)_m (c)_m \frac{(xt(1-t))^m}{(b+c)_m\, m!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c)_n (b)_n}{(b+c+m)_n} \frac{t^n}{n!} (1-t)^{-m} \frac{(b)_m (c)_m (xt)^m (1-t)^m}{(b+c)_m\, m!}$$

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$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (b)_n (c)_n (b)_m (c)_m (xt)^m \cdot t^n}{(b+c)_{n+m} \; n! \; m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (b)_n (c)_n (b)_m (c)_m \, x^m t^{n+m}}{(b+c)_{n+m} \; n! \; m!}$$

by using Lemma

$$\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m, n-m)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^n (b)_{n-m} (c)_{n-m} (b)_m (c)_m}{(b+c)_n \; (n-m)! \; m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m (-1)^m \overset{(b)_n}{\cancel{(1-b-n)_m}}}{(1-n+b)_m} \; \frac{(-1)^n \overset{(c)_n}{\cancel{(1-c-n)_m}}}{(1-n+c)_m}$$

$$\cdot \frac{(b)_n (c)_m x^m t^n (-n)_m (-1)^m}{(b+c)_n \; n! \; m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-n)_m (b)_m (c)_m}{(1-b-n)_m (1-c-n)_m} \frac{x^m}{m!} \right) \frac{(b)_n (c)_n}{(b+c)_n} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_3F_2 \left[\begin{matrix} -n, b, c; \\ 1-b-n, 1-c-n; \end{matrix} \; x \right] \frac{(b)_n (c)_n}{(b+c)_n} \frac{t^n}{n!} \quad \text{—— } (2)$$

From (1) & (2)

$${}_3F_2 \left[\begin{matrix} -n, b, c; \\ 1-b-n, 1-c-n; \end{matrix} \; x \right] = {}_3F_2 \left[\begin{matrix} -\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}, 1-b-c-n \\ 1-b-n, 1-c-n; \end{matrix} \; \frac{-4x}{(1-x)^2} \right]$$

Theorem 30: (Page 87)

If n is a non-negative integer and if a & b are independent of n then

$$ {}_3F_2\left[\begin{matrix} -n, a+n; \tfrac{1}{2}+\tfrac{a}{2}-b; \\ 1+a-b, \tfrac{1}{2}a+\tfrac{1}{2}; \end{matrix} 1\right] = \frac{(b)_n}{(1+a-b)_n} $$

Proof: By using theorem on Page (67)

$$ {}_2F_1\left[\begin{matrix} a, b; \\ 1+a-b; \end{matrix} z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{a}{2}-b; \\ 1+a-b; \end{matrix} \frac{-4z}{(1-z)^2}\right] $$

Then applying the definition on both sides

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1+a-b)_n} \frac{z^n}{n!} = \left[\sum_{k=0}^{\infty} \frac{(\tfrac{1}{2}a)_k (\tfrac{1}{2}+\tfrac{a}{2}-b)_k}{(1+a-b)_k\, k!}\left(\frac{-4z}{(1-z)^2}\right)^k\right](1-z)^{-a} $$

$$ = \sum_{k=0}^{\infty} \frac{(\tfrac{1}{2}a)_k (\tfrac{1}{2}+\tfrac{a}{2}-b)_k (-1)^k \cdot 2^{2k} \cdot z^k}{k!\,(1+a-b)_k\,(1-z)^{2k+a}} $$

Now using $(1-t)^{\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!}$ on $(1-z)^{2k+a}$

$$ = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\tfrac{a}{2})_k (\tfrac{1}{2}+\tfrac{a}{2}-b)_k (-1)^k 2^{2k} \cancel{(\alpha)_{2k}} (2k+a)_n z^{n+k}}{k!\,n!\,(1+a-b)_k} $$

Now using $(a+2k)_n = \frac{(a)_{n+2k}}{(a)_{2k}}$; $(a)_{n+k} = (a)_k (a+n)_k$

$$ = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k (\tfrac{a}{2})_k (\tfrac{1}{2}+\tfrac{a}{2}-b)_k\, 2^{2k} \cdot (a)_{n+2k} \cdot z^{n+k}}{k!\,n!\,(1+a-b)_k (a)_{2k}} . $$

Since $(\alpha)_{2k} = 2^{2k}\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha+1}{2}\right)_k$

$$ \Rightarrow \left(\frac{\alpha}{2}+\frac{1}{2}\right)_k = \frac{(\alpha)_{2k}}{2^{2k}\left(\frac{\alpha}{2}\right)_k} . $$

$$= \sum_{n,k=0}^{\infty} \frac{(\tfrac12+\tfrac12 a-b)_k (a)_{n+2k} (-1)^k z^{n+k}}{k!\, n!\, (1+a-b)_k (\tfrac12 a+\tfrac12)_k}$$

Now using Lemma

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k).$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (\tfrac12+\tfrac12 a-b)_k (a)_{n+k} z^n}{k!\,(n-k)!\,(1+a-b)_k (\tfrac12 a+\tfrac12)_k}$$

Now using

$$(a)_{n+k} = (a)_n (a+n)_k \text{ and } (n-k)! = \frac{(-1)^k n!}{(-n)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k (\tfrac12+\tfrac12 a-b)_k (a+n)_k (a)_n z^n}{k!\,(1+a-b)_k (\tfrac12 a+\tfrac12)_k \cdot n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-n)_k (\tfrac12+\tfrac12 a-b)(a+n)_k}{(1+a-b)_k (\tfrac12 a+\tfrac12)_n k!} \right) \frac{(a)_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} {}_3F_2 \left[\begin{matrix} -n, & a+n, & \tfrac12+\tfrac12 a-b; \\ 1+a-b, & \tfrac12 a+\tfrac12; & \end{matrix} \; 1 \right] \frac{(a)_n z^n}{n!}$$

$\Rightarrow$

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n \cdot z^n}{(1+a-b)_n\, n!} = \sum_{n=0}^{\infty} {}_3F_2 \left[\begin{matrix} -n, a+n, \tfrac12 a+\tfrac{a}{2}-b; \\ 1+a-b, \tfrac12 a+\tfrac12; \end{matrix} \; 1 \right] \frac{(a)_n (z)^n}{n!}$$

$$\Rightarrow {}_3F_2 \left[\begin{matrix} -n, & a+n, & \tfrac12+\tfrac{a}{2}-b; \\ 1+a-b, & \tfrac12 a+\tfrac12; & \end{matrix} \; 1 \right] = \frac{((b)_n)}{(1+a-b)_n}$$

$\Rightarrow$

$${}_3F_2 \left[\begin{matrix} -n, & a+n, & \tfrac12+\tfrac{a}{2}-b; \\ 1+a-b, & \tfrac12 a+\tfrac12; & \end{matrix} \; 1 \right] = \frac{(b)_n}{(1+a-b)_n}$$

Hence Proved.

Theorem: 32)

If neither $(a-b)$ nor $(a-c)$, nor 'a' is a negative integer Then Prove that

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix} x\right] = (1-x)^{-a} {}_3F_2\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}, 1+a-b-c; \\ 1+a-b, 1+a-c; \end{matrix} \frac{-4x}{(1-x)^2}\right] $$

Proof: $$ {}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix} x\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m x^m}{m!(1+a-b)_m (1+a-c)_m} $$

using definition $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and multiplying & dividing $(1+a)_{2m} = \dfrac{\Gamma(1+a+2m)}{\Gamma(1+a)}$ & by $\Gamma(1+a-b-c)$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m x^m}{m!\,(1+a)_{2m}} \cdot \frac{\Gamma(1+a+2m)\Gamma(1+a-b-c)}{\Gamma(1+a-b+m)\,\Gamma(1+a-c+m)} $$

Now again using $\cancel{\Gamma(1+a+2m) = (1+a)}$

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{m=0}^{\infty} {}_2F_1\left[\begin{matrix} b+m, c+m; \\ 1+a+2m; \end{matrix} 1\right] \cdot \frac{(a)_m (b)_m (c)_m x^m}{m!\,(1+a)_{2m}} $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(b+m)_n\,(c+m)_n}{(1+a+2m)_n \cdot n!} \cdot \frac{(a)_m (b)_m (c)_m x^m}{m!\,(1+a)_{2m}} $$

Since $(b+m)_n (a)_m = (b)_{n+m}$, $(c+m)_n (c)_m = (c)_{n+m}$ and $(1+a+2m)_n (1+a)_{2m} = (1+a)_{n+2m}$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n,m=0}^{\infty} \frac{(a)_m (b)_{n+m} (c)_{n+m} x^m}{m!\, n!\, (1+a)_{n+2m}}$$

Now using Lemma

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m).$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_m (b)_n (c)_n x^m}{m!\,(n-m)!\,(1+a)_{n+m}}$$

using $(n-m)! = \dfrac{(-1)^m n!}{(-n)_m}$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^m (-n)_m}{n!\,m!} \frac{(a)_m (b)_n (c)_n x^m}{(1+a)_{n+m}}$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-n)_m (a)_m x^m (-1)^m}{m!\,(1+a+n)_m} \cdot \frac{(b)_n (c)_n x^n}{n!\,(1+a)_n}$$

$(1+a)_{n+m} = (1+a+n)_m (1+a)_n$

using $(1+a)_{n+m} = (1+a+n)_m (1+a)_n$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, a; \\ 1+a+n \end{matrix} -x\right] \cdot \frac{(b)_n (c)_n}{n!\,(1+a)_n}$$

Now using result

$${}_2F_1\left[\begin{matrix} -n, a; \\ 1+a+n; \end{matrix} -x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} \tfrac{1}{2}a, \ \tfrac{1}{2}a+\tfrac{1}{2}; \\ 1+a+n; \end{matrix} \ \frac{-4x}{(1-x)^2}\right]$$

So above relation becomes.

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2} \\ 1+a+n \end{matrix}\ \frac{-4x}{(1-x)^2}\right] \cdot \frac{(b)_n (c)_n}{n!\,(1+a)_n (1-x)^a}$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{n,m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}a)_m (\tfrac{1}{2}a+\tfrac{1}{2})_m\, 2^{2m} x^m}{(1+a+n)_m \cdot m!\,(1-x)^{2m}} \frac{(b)_n (c)_n}{n!\,(1+a)_n (1-x)^a}$$

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!}$$

$$\because\ (\alpha)_{2m} = 2^{2m} \left(\tfrac{\alpha}{2}\right)_m \left(\tfrac{\alpha+1}{2}\right)_m \ \&\ (1+a+n)_m (1+a)_n = (1+a)_{n+m}$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{m,n=0}^{\infty} \frac{(-1)^m (a)_{2m} (b)_n (c)_n x^m}{n!\,m!\,(1+a)_{n+m} (1-x)^{a+2m}}$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+a+m)_n} \frac{1}{n!} \cdot \frac{(-1)^m (a)_{2m} x^m}{m!\,(1+a)_m (1-x)^{a+2m}}$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{m=0}^{\infty} {}_2F_1\left[\begin{matrix} b,\ c; \\ 1+a+m; \end{matrix}\ 1\right] \cdot \frac{(-1)^m (a)_{2m} x^m}{m!\,(1+a)_m (1-x)^{a+2m}}$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \sum_{m=0}^{\infty} \frac{\Gamma(1+a+m)\,\Gamma(1+a-b-c+m)}{\Gamma(1+a-b+m)\,\Gamma(1+a-c+m)} \cdot \frac{(-1)^m (a)_{2m} x^m}{m!\,(1+a)_m (1-x)^{a+2m}}$$

$$= (1-x)^{-a} \sum_{m=0}^{\infty} \frac{\Gamma(1+a+m)}{\Gamma(1+a)} \cdot \frac{\Gamma(1+a-b-c+m)}{\Gamma(1+a-b-c)} \cdot \frac{\Gamma(1+a-b)}{\Gamma(1+a-b+m)} \cdot \frac{\Gamma(1+a-c)}{\Gamma(1+a-c+m)} \cdot \frac{(-1)^m (a)_{2m} x^m}{m!\,(1+a)_m (1-x)^{2m}}$$

$$= \sum_{m=0}^{\infty} \frac{(1+a)_m (1+a-b-c)_m}{(1+a-b)_m (1+a-c)_m (1+a)_m} \frac{(-1)^m (a)_{2m} x^m}{m! (1-x)^{2m}} (1-x)^{-a}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (a)_{2m} (1+a-b-c)_m x^m}{m! (1+a-b)_m (1+a-c)_m (1-x)^{2m}} (1-x)^{-a}$$

$$\therefore = \sum_{m=0}^{\infty} (1-x)^{-a} \frac{(-1)^m \cdot 2^{2m} (a/2)_m \left(\frac{a+1}{2}\right)_m (1+a-b-c)_m x^m}{m! (1+a-b)_m (1+a-c)_m (1-x)^{2m}}$$

$$= (1-x)^{-a} \cdot {}_3F_2 \left[\begin{matrix} a/2, & a/2+1/2, & 1+a-b-c; \\ 1+a-b, & 1+a-c; \end{matrix} \; \frac{-4x}{(1-x)^2} \right]$$

Hence

$${}_3F_2 \left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c \end{matrix} \; x \right] = (1-x)^{-a} \cdot$$

$${}_3F_2 \left[\begin{matrix} a/2, & a/2+1/2, & 1+a-b-c; \\ 1+a-b, & 1+a-c; \end{matrix} \; \frac{-4x}{(1-x)^2} \right]$$

Hence Proved.

Dixon's Theorem:

The series

$${}_{q+1}F_q \left[\begin{matrix} a_0, a_1, a_2, \ldots a_q; \\ b_1, b_2, \ldots, b_q; \end{matrix} \; x \right]$$ is said to be

well posed if

$$1 + a_0 = a_1 + b_1 = a_2 + b_2 = \cdots = a_q + b_q$$

Theorem: 33 (Dixon's Theorem)

The following is an identity if a, b and c are so restricted that each of the functions involved exists.

$$ {}_3F_2\left[\begin{matrix} a, b, c ; \\ 1+a-b, 1+a-c ; \end{matrix} \; 1\right] $$

$$ = \frac{\Gamma(1+a/_2)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a/_2-b-c)}{\Gamma(1+a)\Gamma(1+a/_2-b)\Gamma(1+a/_2-c)\Gamma(1+a-b-c)} $$

Proof: Let $\Psi = {}_3F_2\left[\begin{matrix} a, b, c ; \\ 1+a-b, 1+a-c ; \end{matrix} \; 1\right]$

Then

$$ \Psi = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(1+a-b)_m (1+a-c)_m} \cdot \frac{1}{m!} $$

$$ = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m \Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a-b+m)\Gamma(1+a-c+m)} \cdot \frac{1}{m} $$

Multiplying & dividing by

$$ = (1+a)_{2k} = \frac{\Gamma(1+a+2k)}{\Gamma(1+a)} \text{ \& } \Gamma(1+a-b- $$

$$ \Psi = \sum_{m=0}^{\infty} \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \cdot \frac{\Gamma(1+a+2m)\Gamma(1+a}{m!\,\Gamma(1+a-b+m)\Gamma(1+a-c} $$

$$ (a)_m (b)_m (c)_m $$

But

$$ \frac{\Gamma(1+a+2m)\Gamma(1+a-b-c)}{\Gamma(1+-b+m)\Gamma(1+a-c+m)} = {}_3F_2\left[\begin{matrix} b+m, c+m ; \\ 1+a+2m ; \end{matrix} \; 1\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(b+m)_n (c+m)_n}{(1+a+2m)_n \, n!} $$

Hence Ψ becomes

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$$\Psi = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n,m=0}^{\infty} \frac{(a)_m (b)_{n+m} (c)_{n+m}}{m!\, n!\, (1+a)_{n+2m}}$$

Now using Lemma

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m) \quad .$$

$$\Psi = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n,m=0}^{\infty} \frac{(a)_m (b)_n (c)_n}{m!\,(n-m)!\,(1+a)_{n+m}}.$$

$$\because \quad (n-m)! = \frac{(-1)^m n!}{(-n)_m}$$

$$\Psi = \text{''} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-n)_m (-1)^m (a)_m}{m!\, n!\, (1+a+n)_m} \cdot \frac{(b)_n (c)_n}{(1+a)_n}$$

$$\Psi = \text{''} \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, & a \\ 1+a+n; & \end{matrix} \; -1\right] \frac{(b)_n (c)_n}{n!\,(1+a)_n}$$

Now using theorem

$${}_2F_1\left[\begin{matrix} -n, & a; \\ 1+a+n; & \end{matrix} \; -1\right] = \frac{\Gamma(1+a+n)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a+n)\Gamma(1+a)}$$

$$= \frac{(1+a)_n}{(1+\frac{1}{2}a)_n}$$

$$\Psi = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(1+a)_n (b)_n (c)_n}{(1+\frac{1}{2}a)_n (1+a)_n\, n!}$$

$$\Psi = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+\frac{1}{2}a)_n} \cdot \frac{1}{n!}$$

$$\Psi = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \cdot {}_2F_1\left[\begin{matrix} b, c; \\ 1+\frac{1}{2}a; \end{matrix} \; 1\right]$$

Now using

$${}_2F_1\left[\begin{matrix} a, & b; \\ & c; \end{matrix} \; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

=>

$$\psi = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \cdot \frac{\Gamma(1+\tfrac{1}{2}a)\cdot\Gamma(1+\tfrac{1}{2}\text{[illegible]}}{\Gamma(\tfrac{1}{2}+a-b)\,\Gamma(\tfrac{1}{2}+\text{[illegible]}}$$

Hence

$${}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix} \; 1\right].$$

$$= \frac{\Gamma(1+\tfrac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\tfrac{1}{2}a-b-\text{[illegible]}}{\Gamma(1+a)\,\Gamma(1+\tfrac{1}{2}a-b)\,\Gamma(1+\tfrac{1}{2}a-c)\,\Gamma(1+a-b\text{[illegible]}}$$

Hence Proved.

Date:

Exercise Chapter 5

1) Show that

$${}_0F_1\left[\begin{matrix}-; \\ a;\end{matrix}\ x\right]{}_0F_1\left[\begin{matrix}-; \\ b;\end{matrix}\ x\right] = {}_2F_3\left[\begin{matrix}\tfrac{1}{2}a+\tfrac{1}{2}b,\ \tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2} \\ a,\ b,\ a+b-1;\end{matrix}\ 4x\right]$$

Solution: Consider the Product

$${}_0F_1\left[\begin{matrix}-; \\ a;\end{matrix}\ x\right]{}_0F_1(-; b; x) = \sum_{n,k=0}^{\infty} \frac{x^{n+k}}{(a)_k (b)_n\, k!\, n!}$$

using Lemma

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{x^n}{(a)_k (b)_{n-k}\, k!\, (n-k)!}$$

since $(b)_{n-k} = \dfrac{(-1)^k (b)_n}{(1-b-n)_k}$ & $(n-k)! = \dfrac{(-1)^k n!}{(-n)_k}$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1-b-n)_k}{(-1)^k (b)_n} \cdot \frac{(-1)^k(-n)_k}{n!} \cdot \frac{x^n}{(a)_k\, k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1-b-n)_k\, (-n)_k}{(a)_k\, k!} \cdot \frac{x^n}{(b)_n\, n!}$$

$$= \sum_{n=0}^{\infty} F\left[\begin{matrix}-n,\ 1-b-n; \\ a;\end{matrix}\ 1\right] \frac{x^n}{(b)_n\, n!}$$

Now using result

$$F\left[\begin{matrix}-n,\ 1-b-n; \\ a;\end{matrix}\ 1\right] = \frac{(a+b-1)_{2n}}{(a)_n\, (a+b-1)_n}$$

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$$=\sum_{n=0}^{\infty} \frac{(a+b-1)_{2n}}{(a)_n (a+b-1)_n} \cdot \frac{x^n}{(b)_n n!}$$

$$(a+b-1)_{2n} = 2^{2n} \left(\frac{a+b-1}{2}\right)_n \left(\frac{a+b-1+1}{2}\right)_n$$

$$= 2^{2n} \left(\frac{a+b-1}{2}\right)_n \left(\frac{a+b}{2}\right)_n$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{a+b-1}{2}\right)_n \left(\frac{a+b}{2}\right)_n \cdot 2^{2n} x^n}{(a)_n \, (a+b-1)_n \, (b)_n \, n!}$$

$$= {}_2F_3 \left[\begin{matrix} \frac{a+b}{2}, \ \frac{a+b-1}{2} ; \\ a, \ b, \ a+b-1 ; \end{matrix} \ 4x \right]$$

Hence Proved.

Problem 2: Show that

$$\int_0^t x^{1/2} (t-x)^{-1/2} \left[1 - x^2 (t-x)^2\right]^{-1/2} dx = \frac{1}{2} \pi t \, {}_2F_1 \left[\begin{matrix} 1/4, \ 3/4 ; \\ 1 ; \end{matrix} \ \frac{t^4}{16} \right]$$

Solution:

L.H.S $\displaystyle\int_0^t x^{1/2} (t-x)^{-1/2} \left[1 - x^2 (t-x)^2\right]^{-1/2} dx.$

since $\displaystyle (1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n (y^n)}{n!}$

So $\displaystyle \left[1 - x^2 (t-x)^2\right]^{-1/2} = \sum_{n=0}^{\infty} (1/2)_n \frac{\left[x^2 (t-x)^2\right]^n}{n!}$

$$= {}_1F_0 \left[1/2 ; - ; x^2 (t-x)^2 \right]$$

So

$$L.H.S = \int_0^t x^{1/2} (t-x)^{-1/2} \, {}_1F_0 \left(1/2 ; - , x^2 (t-x)^2 \right) dx.$$

Now using the result

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$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\, {}_pF_q(a_1,a_2,\dots a_p; b_1,b_2,\dots b_q; c x^m (t-x)^s)\,dx.$$
$$= B(\alpha,\beta)\, t^{\alpha+\beta-1}\, {}_{p+m+s}F_{q+m+s}$$
$$\left(a_1,a_2,\dots,a_p, \frac{\alpha}{m}, \frac{\alpha+1}{m}, \dots \frac{\alpha+m-1}{m}, \frac{\beta}{s}, \frac{\beta+1}{s}, \dots \frac{\beta+s-1}{s};\right.$$
$$\left. b_1, b_2, \dots b_q, \frac{\alpha+\beta}{m+s}, \frac{\alpha+\beta+1}{m+s}, \dots \frac{\alpha+\beta+m+s-1}{m+s}; \frac{m^m s^s c t^{m+s}}{(m+s)^{m+s}}\right)$$

Now Comparing L.H. of this with the above L.HS.

$\alpha-1 = 1/2 \Rightarrow \alpha = 3/2$, $\beta-1 = -1/2 \Rightarrow \beta = 1/2$, $P=1$, $q=0$

$m=2$, $s=2$, $c=1$, So

$$\int_0^1 x^{1/2}(t-x)^{-1/2}\, {}_1F_0\left(1/2; -; x^2(1-t)^2\right) dx.$$
$$= B(3/2, 1/2)\, t^{3/2+1/2-1}\, {}_5F_4$$
$$\left[\frac{1}{2}, \frac{3/2}{2}, \frac{3/2+1}{2}, \frac{1/2}{2}, \frac{1/2+1}{2}; -; \frac{3/2+1/2}{4}, \frac{3/2+1/2+1}{4}, \frac{3/2+1/2+2}{4}, \frac{3/2+1/2+3}{4}; \frac{2^2 2^2 t^4}{(2+2)^4}\right]$$
$$= \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(3/2+1/2)}\, t\, {}_5F_4\left(1/2, 3/4, 5/4, 1/4, 3/4; -, 1/2, 3/4, 1, 5/4; \frac{t^4}{16}\right)$$
$$= \frac{\Gamma(1/2+1)\Gamma(1/2)}{\Gamma(2)}\, t \sum_{n=0}^{\infty} \frac{(1/2)_n (3/4)_n (5/4)_n (1/4)_n (3/4)_n}{(1/2)_n (3/4)_n (1)_n (5/4)_n\, n!}\left(\frac{t^4}{16}\right)^n$$
$$= 1/2\, \Gamma(1/2)\Gamma(1/2)\, t \sum_{n=0}^{\infty} \frac{(1/4)_n (3/4)_n}{(1)_n\, n!}\left(\frac{t^4}{16}\right)^n$$
$$\because \Gamma(2) = 1, \ \Gamma(1/2) = \sqrt{\pi}.$$
$$= 1/2\, \sqrt{\pi}\cdot\sqrt{\pi}\, t\, F\left(1/4, 3/4; 1; \frac{t^4}{16}\right)$$
$$= 1/2\, \pi t\, F\left(1/4, 3/4; 1; t^4/16\right)$$

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