Chapter 5:

Generalized Hyper Geometric Functions

The generalized Hyper Geometric function is defined as

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\prod_{r=1}^{p} (a_r)_n \Big/ \prod_{s=1}^{q} (b_s)_n \right) \cdot \frac{z^n}{n!}$$

in which no denominator parameter b_s is zero or a negative integer. If any a_p is zero or negative integer then the series terminates.

P = Number of factors in Numerator
q = Number of factors in Denominator

Properties: (a) If $P \leq q$, the series converges for all finite z.

If a's and b's are equal and z is finite then this is cgt. If a's < b's and z is finite then it is convergent.

(b) If $P = q+1$ the series converges for $|z|<1$ and diverges for $|z|>1$

If $P = q+1$ i.e. If $P=4$, $q=3$ Denominator term is one less than Numerator and $|z|<1$ then it is cgt. and if $|z|>1$ then it diverges

${}_2F_1$
${}_3F_2$
$\vdots$
${}_{m+1}F_m$

(c) If $P > q+1$ the series diverges for $z \neq 0$

If $P=5$, then $q=3$ at least 2 less than P then for $z \neq 0$ the series always diverges.

* If the series terminates there is no question of convergence and the conclusion (b) & (c) do not apply.

* If it does not terminates then it is dgt.
* A finite series is always converged

If $p = q+1$, then the series in (2) is absolutely convergent on the circle $|z| = 1$ if

$$Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$$

* If both p & q or any one of them is zero then we use the — dash notation for the absence of parameters.

e.g $\;{}_0F_1(-;b;z) = \sum_{n=0}^{\infty} \frac{z^n}{(b)_n n!}$

or $\;{}_0F_0(-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

The Exponential and Binomial Functions:

$${}_0F_0(-;-;z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z)$$

If p & q both are zero

If we use one numerator parameter and no denominator parameter then

$${}_1F_0(a;-;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = (1-z)^{-a}$$

A Differential Equation: Previously ordinary hypergeometric function $F(a,b,c,z)$ satisfies DE

$z(1-z)\frac{d^2w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0$ or in terms of diff. operator $\theta = z\frac{d}{dz}$, DE is

$[\theta(\theta+c-1) - z(\theta+a)(\theta+b)]w = 0$

Now we will develope DE using generalized Hypergeometric function

$$w = {}_pF_q = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

Since $\theta z^k = z\frac{d}{dz}z^k = zkz^{k-1} = kz^k$

So

$$\theta \prod_{j=1}^{q} (\theta + b_j - 1) w = \sum_{k=0}^{\infty} k \frac{\prod_{j=1}^{q} (k + b_j - 1) \prod_{i=1}^{p} (a_i)_k}{\prod_{j=1}^{q} (b_j)_k} \cdot \frac{z^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \frac{(k+b_1-1)(k+b_2-1) \cdots (k+b_q-1) \prod_{i=1}^{p} (a_i)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k}$$

$$= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \frac{\prod_{i=1}^{p} (a_i)_k \, (\cancel{k+b_1-1})(\cancel{k+b_2-1}) \cdots (\cancel{k+b_q-1})}{[b_1(b_1+1) \cdots (\cancel{b_1+k-1})][b_2(b_2+1) \cdots (\cancel{b_2+k-1})] \cdots [b_q(b_q+1) \cdots (\cancel{b_q+k-1})]}$$

$$= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \frac{\prod_{i=1}^{p} (a_i)_k}{[b_1(b_1+1) \cdots (b_1+(k-1)-1)][b_2(b_2+1) \cdots (b_2+(k-1)-1)] \cdots [b_q(b_q+1) \cdots (b_q+(k-1)-1)]}$$

$$= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \frac{\prod_{i=1}^{p} (a_i)_k}{(b_1)_{k-1} (b_2)_{k-1} \cdots (b_q)_{k-1}}$$

Replace k by $k+1$.

$$= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!} \frac{\prod_{i=1}^{p} (a_i)_{k+1}}{\prod_{j=1}^{q} (b_j)_k}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)(a_i+1) \cdots (a_i+(k+1)-1)}{\prod_{j=1}^{q} (b_j)_k} \cdot \frac{z^{k+1}}{k!}$$

$$= \sum_{k=0}^{\infty} \prod_{i=1}^{p} (a_i+k) \frac{\prod_{i=1}^{p} (a_i)_k}{\prod_{j=1}^{q} (b_j)_k} \frac{z^{k+1}}{k!}$$

$$= z \prod_{i=1}^{p} (a_i + \theta) w$$

$$\Rightarrow \theta \prod_{j=1}^{q} (\theta + b_j - 1) w = z \prod_{i=1}^{p} (a_i + \theta) w$$

$$\Rightarrow \left[\theta \prod_{j=1}^{q} (\theta + b_j - 1) - z \prod_{i=1}^{p} (a_i + \theta)\right] w = 0$$

where no b_j is a non-positive integer

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A Simple Integral:

Theorem: If $p \le q+1$ and $Re(b_1) > Re(a_1) > 0$, there is no one $b_1, b_2, \ldots, b_q$ is zero or non-negative integer, $|z|<1$, then

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots a_p; \\ b_1, b_2, \ldots b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} \, {}_{p-1}F_{q-1}\left[\begin{matrix} a_2, a_3, \ldots, a_p; \\ b_2, b_3, \ldots, b_q; \end{matrix} zt\right] dt $$

Proof: Consider.

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots a_p; \\ b_1, b_2, \ldots b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \quad (1) $$

$$ = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)}{\Gamma(a_1)} \cdot \frac{\Gamma(b_1)}{\Gamma(b_1+n)} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

$$ \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\Gamma(b_1-a_1)}{\Gamma(b_1+n)} \cdot \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

$$ = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \sum_{n=0}^{\infty} \left[\int_0^1 t^{a_1+n-1} (1-t)^{b_1-a_1-1} dt \right] \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

Since the series (1) is uniformly convergent, therefore.

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots a_p; \\ b_1, b_2, \ldots b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} \left(\sum_{n=0}^{\infty} \frac{(a_2)_n \cdots (a_p)_n}{(b_2)_n \cdots (b_q)_n} \frac{(zt)^n}{n!} \right) dt $$

$$ = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} \, {}_{p-1}F_{q-1}\left[\begin{matrix} a_2, a_3, \ldots, a_p; \\ b_2, b_3, \ldots b_q; \end{matrix} zt\right] dt $$

(A simple Integral form of G.H.F by Driver & Thorston 2006)

As Required.

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Theorem: If $Re(c) > Re(b) > 0$, $|z| < 1$, then.

$${}_{m+1}F_m\left[\begin{matrix} a, \frac{b}{m}, \frac{b+1}{m}, \ldots, \frac{b+m-1}{m}; \\ \frac{c}{m}, \frac{c+1}{m}, \ldots, \frac{c+m-1}{m}; \end{matrix} z\right]$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^m)^{-a}\,dt$$

Proof: ${}_{m+1}F_m\left[\begin{matrix} a, \frac{b}{m}, \frac{b+1}{m}, \ldots \frac{b+m-1}{m}; \\ \frac{c}{m}, \frac{c+1}{m}, \ldots \frac{c+m-1}{m}; \end{matrix} z\right]$

$$= \sum_{n=0}^{\infty} \frac{(a)_n \left(\frac{b}{m}\right)_n \left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-1}{m}\right)_n}{\left(\frac{c}{m}\right)_n \left(\frac{c+1}{m}\right)_n \cdots \left(\frac{c+m-1}{m}\right)_n} \cdot \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_{mn}}{(c)_{mn}} \cdot \frac{z^n}{n!} \qquad (b)_{mn} = m^{mn}\left(\frac{b}{m}\right)_n\left(\frac{b+1}{m}\right)_n \cdots \left(\frac{b+m-1}{m}\right)_n$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+mn)}{\Gamma(c+mn)} \cdot (a)_n \frac{z^n}{n!} \qquad \text{since } (b)_{mn} = \frac{\Gamma(b+mn)}{\Gamma(b)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+mn)\Gamma(c-b)}{\Gamma(c+mn)} (a)_n \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \left[\int_0^1 t^{b+mn-1}(1-t)^{c-b-1}\,dt\right] \frac{(a)_n z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left[\sum_{n=0}^{\infty} \frac{(zt^m)^n (a)_n}{n!}\right] dt$$

$$\because (1-t)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^m)^{-a}\,dt$$

(Some Results on ${}_mF_m$ Shahid Mabeen)

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