Principal axis make an angle of $\pi/4$ with coordinate axis i.e principal axis in this case are along the diagonals of the square,

Now using, $I_{x'x'} = A\cos^2\theta + B\sin^2\theta + F\sin 2\theta$

$$= \frac{5ma^2}{2}\left(\frac{1}{2}\right) + \frac{5ma^2}{2}(1/2) + \frac{ma^2}{2}(1)$$

$$= \frac{12ma^2}{4} \quad (\text{by lcm})$$

$$= 3ma^2$$

Similarly

$$I_{y'y'} = A\sin^2\theta + B\cos^2\theta - F\sin 2\theta$$

$$= 2ma^2$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = 3ma^2 + 2ma^2 = 5ma^2.$$

Equimomental Systems:-

Two systems are said to be equimomental systems if they have same (equal) moments of inertia about every line in space.

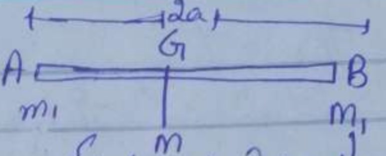
Theorem:-

Two systems are equimomental. if
i) they have same total mass,
ii) They have same principal axis
iii) They have same centroid.

Q:-

Show that a uniform rod of mass m is equimomental to three particles of masses $\frac{m}{6}, \frac{m}{6}$ & $\frac{2m}{3}$ attached first & 2^{nd} at its ends & 3rd at its centre?

Sol:



G.O.C نہ تبدیل ہونے والے
دو Same masses

Consider a rod AB of length $2a$ & mass M. Then its m.i at a line $\perp$ to the rod

$AB = \frac{1}{3} Ma^2 \longrightarrow (1)$

Now, Consider three masses m_1, m_1, m_2 attached at A, B & G resp.

& $m_1 + m_2 + m_1 = M$

$m_2 = M - 2m_1$

Now, m.i of these three masses,

$I = \sum_{i=1}^{3} m_i d_i^2$

$= m_1 d_1^2 + m_2 d_2^2 + m_3 d_3^2$

$= m_1 a^2 + m_1 a^2 + m_2 (0) \Rightarrow 2m_1 a^2.$

The system would be equimomental if $2m_1 a^2 = \frac{1}{3} Ma^2.$

$m_1 = \frac{1}{6} M.$

for $m_2 = M - 2m_1.$

$= M - \frac{M}{3} = \frac{2M}{3}$

i.e for equimomental system masses are, $\frac{M}{6}, \frac{M}{6}$ & $\frac{2M}{6}$.

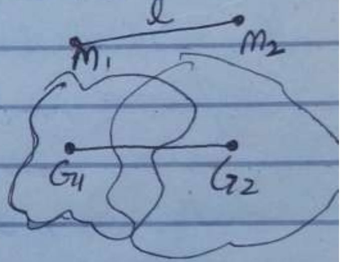
Note $I = M/6 \cdot a^2 + \frac{M}{6} a^2 \neq \frac{2M}{3} \times 0$

Theorem:

The two given systems are equimomental if they have the same total mass ii) same P. axs iii) same Centroid

Proof: Consider m_1 & m_2 are two masses with G_1 & G_2 these C.O.M (centroid)
Join G_1 & G_2 as the two systems are equimomental. i.e They have same M.I about any line in space.
we Consider M.I of the two systems abt the line G_1, G_2 as I.
Draw a line l ∥ to $G_1 G_2$ then by using parallel axis theorem

$$= I + m_1 h^2$$



M.I of 1st system abt $l = I + m_1 h^2$
M.I of 2nd system abt $l = I + m_2 h^2$

∵ Systems are equimomental.

$$I + m_1 h^2 = I + m_2 h^2$$
$$m_1 h^2 = m_2 h^2$$
$$m_1 = m_2 = M \text{ (say)}.$$

Consider two ∥ lines G_1H_1 & G_2H_2 & each of the line, G_1H_1 & G_2H_2 is ⊥ to $G_1 G_2$ also. Then

M.I of 1st system abt line $G_1H_1 = I + M|G_1G_2|^2$
" " 2nd " " $= I + 0$

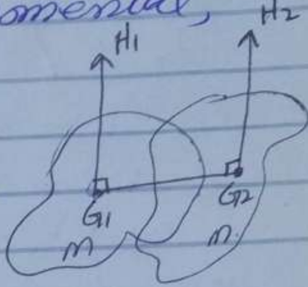
using Conditions of equimomental,

$$I + M|G_1G_2|^2 = I$$

$$M|G_1G_2|^2 = 0$$

$$M \neq 0; \Rightarrow |G_1G_2|^2 = 0$$

$$G_1 = G_2 = G \text{ (say)}$$



also, let the two systems have same M.I about G_2H
∴ the two systems are equimom. i.e they have same M.I about any line, so they have same M.I about principal axis, so there P. axis are same.

Proved

after mid

Euler's Angles:-

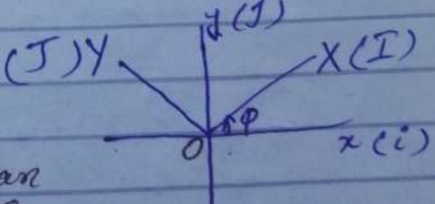
mmmmmmww Wed;

$$oxyz \xrightarrow{abt} oXYz \longrightarrow oX'Y'z' \xrightarrow{abt} ox'y'z'$$

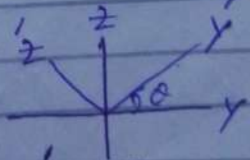
abt rotation not changed.

جو چیز دل سے Seple پوری میں صرف ان کے اینگلز کو rotation like ... ہوتا ہے anticlockwise.

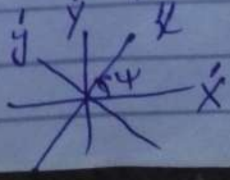
(φ, θ, ψ)



first rotation:- (ϕ) precision angle



2nd rotation:- (θ) nutation angle



3rd rotation:- (ψ) body angle

Motion of a rigid body in space can be described by three angular displacements (φ, θ, ψ), where of the coordinate system $oxyz$ to $oXYZ$; $oXYZ$ to $o\acute{X}\acute{Y}\acute{Z}$ & $o\acute{X}\acute{Y}\acute{Z}$ to $o\acute{x}\acute{y}\acute{z}$ respectively. where, first rotation is about z-axis, 2^{nd} rotation is abt X-axis, & 3^{rd} rotation is abt Z'-axis.

Let, i, j, k ; I, J, K ; $\acute{I}, \acute{J}, \acute{K}$ & $\hat{i}', \acute{j}, \acute{k}$ are the unit vectors along the axis in the respective four coordinate system.

for first rotation about z-axis through an angle φ.

$$i = (i.I)I + (i.J)J + (i.K)K$$

This relation is from;

$$\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\bar{A} = (\bar{A}.\hat{i})\hat{i} + (\bar{A}.\hat{j})\hat{j} + (\bar{A}.k)k. \qquad \left(\because A.i = A_1 \quad \because A.j = A_2 \right.$$

i, j, k کو φ سے Anticlockwise روٹیشن دی ہے

$$i = \cos\varphi\, I + \cos(90+\varphi)J + \cos 90\, K$$

$$= \cos\varphi\, I - \sin\varphi\, J \longrightarrow (1)$$

$$j = (j.I)I + (j.J)J + (j.K)K$$

$$= \cos(90-\varphi)I + \cos\varphi\, J + \cos 90\, K.$$

$$= \sin\varphi\, I + \cos\varphi\, J \longrightarrow (2)$$

$$k = (k.I)I + (k.J)J + (k.K)K.$$

$$= \cos 90\, I + \cos 90\, J + \cos o\, K$$

$$= K \longrightarrow (3)$$

So, first transformation eqs are given by ①, ② & ③

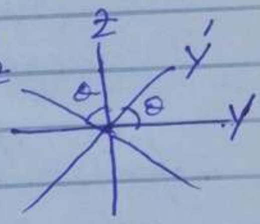
Now

ii) for 2nd rotation abt X-axis through an angle θ; we have,

$I = (I \cdot I')I' + (I \cdot J')J' + (I \cdot K')K'$;

$= \cos(0) I' + \cos 90 J' + \cos 90 K'$

$= I' \longrightarrow ④$



Now

$J = (J \cdot I')I' + (J \cdot J')J' + (J \cdot K')K'$

$= \cos(90) I' + \cos\theta J' + \cos(90+\theta) K'$

$= \cos\theta J' - \sin\theta K' \longrightarrow ⑤$

$K = (K \cdot I')I' + (K \cdot J')J' + (K \cdot K')K'$

$= \cos(90) I' + \cos(90-\theta) J' + \cos\theta K'$

$= \sin\theta J' + \cos\theta K' \longrightarrow ⑥$

So, 2nd transformation eqs are given by ④, ⑤ & ⑥

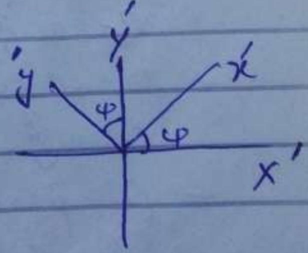
Now

iii) for 3rd rotation abt z' axis through an angle φ.

$I' = (I' \cdot i')i' + (I' \cdot j')j' + (I' \cdot k')k'$;

$= \cos\varphi\, i' + \cos(90+\varphi) j' + \cos 90 k'$

$= \cos\varphi\, i' - \sin\varphi\, j' \longrightarrow ⑦$



$J' = (J' \cdot i')i' + (J' \cdot j')j' + (J' \cdot k')k'$

$= \cos(90-\varphi) i' + \cos\varphi\, j' + \cos 90 k'$

$= \sin\varphi\, i' + \cos\varphi\, j' \longrightarrow ⑧$

$$k' = (k'\cdot i')i' + (k'\cdot j')j' + (k'\cdot k')k'$$
$$= \cos 90\, i' + \cos 90\, j' + \cos(0) k'$$
$$= k' \longrightarrow (9)$$

So,
Transformation eqtn's for 3rd rotation are given by (7), (8) & (9).

Deduction:-

find Transformation eq's from $\hat{i}, \hat{j}, \hat{k}$ into i', j', k'

$$\because \hat{i} = \cos\phi I - \sin\phi J$$
$$= \cos\phi I' - \sin\phi(\cos\theta J' - \sin\theta k')$$
$$= \cos\phi I' - \sin\phi\cos\theta J' + \sin\phi\sin\theta k'$$
$$= \cos\phi(\cos\psi i' - \sin\psi j') - \sin\phi\cos\theta (\sin\psi i' + \cos\psi j') + \sin\phi\sin\theta k')$$
$$= (\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta)i' - (\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta)j' + \sin\phi\sin\theta k' \longrightarrow (A)$$

Now

$$\hat{j} = \sin\phi I + \cos\phi J$$
$$= \sin\phi I' + \cos\phi(\cos\theta J' - \sin\theta k')$$
$$= \sin\phi(\cos\psi i' - \sin\psi j') + \cos\phi[(\cos\theta\sin\psi i' + \cos\theta\cos\psi j' - \sin\theta k']$$
$$= \sin\phi\cos\psi i' - \sin\phi\sin\psi j' + \cos\phi\cos\theta\sin\psi i' + \cos\phi\cos\theta\cos\psi j' - \cos\phi\sin\theta k'$$
$$= (\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta)i' + (\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi)j' - \cos\phi\sin\theta k' \longrightarrow (B)$$

Similarly,

$$\hat{k} = K$$
$$= \sin\theta J' + \cos\theta K'$$
$$= \sin\theta(\sin\psi \hat{i} + \cos\psi \hat{j}) + \cos\theta \hat{k}$$
$$= \sin\theta \sin\psi \hat{i} + \sin\theta\cos\psi \hat{j} + \cos\theta \hat{k} \rightarrow (C)$$

(A), (B), (C) represents required transformation

Alternatively.-

first rotation can be written as,

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ J \\ K \end{bmatrix}$$

$$\bar{x} = R_\phi \bar{X} \rightarrow (1)$$

2nd rotation.

$$I = I'$$
$$J = \cos\theta J' - \sin\theta K'$$
$$K = \sin\theta J' + \cos\theta K'$$

$$\begin{bmatrix} I \\ J \\ K \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} I' \\ J' \\ K' \end{bmatrix}$$

$$\bar{X} = R_\theta \bar{X}' \rightarrow (2)$$

3rd rotation

$$\begin{bmatrix} I' \\ J' \\ K' \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i' \\ j' \\ k' \end{bmatrix}$$

$$\bar{X}' = R_\psi \bar{x}' \rightarrow (3)$$

from ①

$$\bar{x} = R_\phi \bar{X}$$
$$= R_\phi R_\theta \bar{X}'$$
$$= R_\phi R_\theta R_\psi \bar{x}'$$

$$\bar{x} = R\bar{x}'$$

Where, $R = R_\phi R_\theta R_\psi$.

Derived Euler's Geometrical eqs

If $\bar{\omega}'$ is angular velocity after the three rotations.

$\boxed{\hat{k} = K}$ $\boxed{I = I'}$ $\boxed{k = k'}$

$$\bar{\omega}' = \omega_\phi \hat{k} + \omega_\theta \hat{I} + \omega_\psi k'$$
$$= \omega_\phi(\sin\theta j' + \cos\theta k') + \omega_\theta(\cos\psi i' - \sin\psi j') + \omega_\psi k'$$
$$= \omega_\phi[\sin\theta(\sin\psi i' + \cos\psi j') + \cos\theta k'] + \omega_\theta \cos\psi i'$$
$$- \omega_\theta \sin\psi j' + \omega_\psi k'$$
$$= \omega_\phi \sin\theta \sin\psi i' + \omega_\phi \sin\theta \cos\psi j' + \omega_\phi \cos\theta k' + \omega_\theta \cos\psi i'$$
$$- \omega_\theta \sin\psi j' + \omega_\psi k'$$
$$\bar{\omega}' = (\omega_\phi \sin\theta \sin\psi + \omega_\theta \cos\psi) i' + (\omega_\phi \sin\theta \cos\psi - \omega_\theta \sin\psi) j'$$
$$+ (\omega_\phi \cos\theta + \omega_\psi) k'$$

Where,

$$\begin{cases} \omega'_x = \omega_\phi \sin\theta \sin\psi + \omega_\theta \cos\psi = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega'_y = \omega_\phi \cos\psi \sin\theta - \omega_\theta \sin\psi = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \omega'_z = \omega_\phi \cos\theta + \omega_\psi = \dot{\phi}\cos\theta + \dot{\psi} \end{cases}$$

are known as Euler's geometrical eq's,

Euler's Dynamical eqs of motion

let a rigid body is rotating with angular velocity $\bar{\omega}$ abt a fixed point O, let OX, OY & OZ are principal axis, if L & $[I_{ij}]$ are angular momentum & inertia metrix then,

$$[L]=[I][\bar{\omega}]$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

∴ Abt principal axis, we have,

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = I_{zx} = I_{xz} = 0$$

(i.e product of inertia vanishes abt principal axis")

So,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$L_x = I_{xx}\omega_x \quad \text{or} \quad L_1 = I_1\omega_1$$

$$L_y = I_{yy}\omega_y \quad \text{or} \quad L_2 = I_2\omega_2$$

$$L_z = I_{zz}\omega_z \quad \text{or} \quad L_3 = I_3\omega_3$$

So,

$$\bar{L} = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$$

$$\bar{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k} \rightarrow (1)$$

using rotating axis theorem

$$\left(\frac{d\bar{A}}{dt}\right)_f = \left(\frac{d\bar{A}}{dt}\right)_r + \bar{\omega} \times \bar{A}$$

So,

$$\left(\frac{d\bar{L}}{dt}\right)_f = \left(\frac{dL}{dt}\right)_r + \bar{\omega} \times \bar{L} \longrightarrow (2)$$

as, $\underline{L} = \underline{r} \times \underline{p} \longrightarrow$ linear Mom.

$$\left(\frac{dL}{dt}\right)_f = \frac{d\bar{r}}{dt} \times \bar{p} + \bar{r} \times \frac{d\bar{p}}{dt}$$

$$= \bar{r} \times p + \bar{r} \times m\bar{a}$$

$$= 0 + \bar{r} \times \bar{F}$$

$$= \bar{G} \text{ (say)} \longrightarrow (3)$$

$\because p = mv$

$\frac{d}{dt}(mv)$

$\therefore m\frac{dv}{dt}$

$m\bar{a}$

$$\left(\frac{d\bar{L}}{dt}\right)_r = \frac{d}{dt}(I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k})$$

$$= I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} \longrightarrow (4).$$

Now

$$\bar{\omega} \times \bar{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= \omega_2\omega_3(I_3 - I_2)\hat{i} + \omega_1\omega_3(I_1 - I_3)\hat{j} + \omega_1\omega_2(I_2 - I_1)\hat{k} \longrightarrow (5)$$

using values from,
(3), (4) & (5) into eq (2)

$$\bar{G} = I_1\dot{\omega}_1\hat{i} + I_2\dot{\omega}_2\hat{j} + I_3\dot{\omega}_3\hat{k} + \omega_2\omega_3(I_3 - I_2)\hat{i} + \omega_1\omega_3(I_1 - I_3)\hat{j} + \omega_1\omega_2(I_2 - I_1)\hat{k}$$

Now Comparing:

$$\left.\begin{aligned} G_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ G_2 &= I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) \\ G_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{aligned}\right] \rightarrow (6)$$

These are known as Euler dynamical eqts.

Deduction:

for torque free motion $\Rightarrow \bar{G} = 0$
then Euler's eqs becomes,

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0 \rightarrow (7)$$
$$I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0 \rightarrow (8)$$
$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0 \rightarrow (9)$$

xply eq (7) by ω_1, (8) by ω_2, (9) by ω_3

$$I_1\omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3(I_3 - I_2) = 0$$
$$I_2\omega_2\dot{\omega}_2 + \omega_1\omega_2\omega_3(I_1 - I_3) = 0$$
$$I_3\omega_3\dot{\omega}_3 + \omega_1\omega_2\omega_3(I_2 - I_1) = 0$$

adding these, we get,

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0.$$

or $$\frac{1}{2}\frac{d}{dt}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] = 0$$

$\left(\frac{d}{dt}\frac{1}{2}\omega_1^2 = \frac{1}{2}\cdot 2\omega_1\dot{\omega}_1\right)$

Integ.

$$\frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right] = \text{Const.}$$

$$\frac{1}{2}\left[I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3 k\right]\cdot\left[\omega_1\hat{i} + \omega_2\hat{j} + \omega_3 k\right] = c$$

$$\frac{1}{2} I\cdot\bar{\omega} = c \rightarrow (10)$$

This is known as energy theorem.

Q1: Show that for a torque free motion magnitude of angular mom. is Constant.

Sol:

for a torque free motion, Euler's dynamical eqs are,

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = 0 \rightarrow (1)$$
$$I_2 \dot{\omega}_2 - \omega_1 \omega_3 (I_3 - I_1) = 0 \rightarrow (2)$$
$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0 \rightarrow (3)$$

xply (1) $I_1\omega_1$, (2) $I_2\omega_2$ & (3) $I_3\omega_3$

$$I_1^2 \omega_1 \dot{\omega}_1 - \omega_1 \omega_2 \omega_3 (I_1 I_2 - I_1 I_3) = 0$$
$$I_2^2 \omega_2 \dot{\omega}_2 - \omega_1 \omega_2 \omega_3 (I_2 I_3 - I_2 I_1) = 0$$
$$I_3^2 \omega_3 \dot{\omega}_3 - \omega_1 \omega_2 \omega_3 (I_3 I_1 - I_3 I_2) = 0$$

adding all these,

$$I_1^2 \omega_1 \dot{\omega}_1 + I_2^2 \omega_2 \dot{\omega}_2 + I_3^2 \omega_3 \dot{\omega}_3 = 0$$
$$\frac{1}{2} \frac{d}{dt} (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2) = 0$$

Integ.

$$\frac{1}{2} (I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2) = c^*$$
$$(I_1\omega_1 \hat{i} + I_2\omega_2 \hat{j} + I_3 \omega_3 \hat{k}) \cdot (I_1\omega_1 \hat{i} + I_2\omega_2 \hat{j} + I_3 \omega_3 \hat{k}) = c$$
$$L \cdot L = c$$
$$|L^2| = c$$
$$|L| = c$$

$\Rightarrow$ Mag. of A.M is Const.

Q2. In case of spherical torque free motion, angular velocity is Constant.

Sol:

∴ for a torque free motion, Euler's dynamical eqs are,

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0$$
$$I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1) = 0$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0$$

for any spherical top.

$$I_1 = I_2 = I_3 = I \neq 0$$

$$\Rightarrow I_1\dot{\omega}_1 = 0 \; ; \; I\dot{\omega}_2 = 0, \; I\dot{\omega}_3 = 0$$

integ. all,

$$\Rightarrow \omega_1 = c_1$$
$$\Rightarrow \omega_2 = c_2$$
$$\Rightarrow \omega_3 = c_3$$

therefore;

$$\omega = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} = \text{Const.}$$

$\Rightarrow$ Angular vel is Const.

Q3:

An ellipsoid torque free motion about its centre, set into motion at $t=0$ & $(n, 0, 3n)$ as components of its angular velocity $\bar{\omega}$ along the principal axis, $6A, 3A$ & A are M.I about the axis,

find angular velocity at any time t & show that when $t \to \infty$ the mag. of the angular velocity is $n\sqrt{5}$

Sol:

for forque free motion, Euler's eq of motion are, $I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = 0 \rightarrow (1)$

$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \rightarrow (2)$

$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \rightarrow (3)$

using $I_1 = 6A,\ I_2 = 3A,\ I_3 = A$

We have,

$6A\dot{\omega}_1 - 2A\omega_2\omega_3 = 0$ ÷ by 2

$3A\dot{\omega}_1 - A\omega_2\omega_3 = 0 \rightarrow (4)$

$3A\dot{\omega}_2 + 5A\omega_1\omega_3 = 0 \rightarrow (5)$

$A\dot{\omega}_3 - 3A\omega_1\omega_2 = 0 \rightarrow (6)$

$(4)\ 5\omega_1 \Rightarrow \quad 15A\omega_1\dot{\omega}_1 - 5A\omega_1\omega_2\omega_3 = 0$

$(5)\ \omega_2 \Rightarrow \quad 3A\omega_2\dot{\omega}_2 + 5A\omega_1\omega_2\omega_3 = 0$

$15A\omega_1\dot{\omega}_1 + 3A\omega_2\dot{\omega}_2 = 0$

$3A(5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2) = 0$

$5\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$

Integ

$\frac{5\omega_1^2}{2} + \frac{\omega_2^2}{2} = c_1'$

$5\omega_1^2 + \omega_2^2 = 2c_1' \quad (\because 2c_1' = c_1$

$5\omega_1^2 + \omega_2^2 = c_1 \rightarrow (7)$

at $t = 0,\ \omega_1 = n,\ \omega_2 = 0$

$5n^2 = c_1$

$5\omega_1^2 + \omega_2^2 = 5n^2$

$5\omega_1^2 = 5n^2 - \omega_2^2$

$\omega_1 = \sqrt{\frac{5n^2 - \omega_2^2}{5}} \rightarrow (8)$

from (4) & (6)

(4) $3\omega_1$, (6) ω_3.

$$9A\dot{\omega}_1\omega_1 - 3A\omega_1\cancel{\omega_2}\omega_3 = 0$$
$$-A\dot{\omega}_3\omega_3 \mp 3A\omega_1\omega_2\omega_3 = 0$$
$$9A\dot{\omega}_1\omega_1 - A\dot{\omega}_3\omega_3 = 0$$

$$9\dot{\omega}_1\omega_1 - \dot{\omega}_3\omega_3 = 0$$

integ.

$$9\frac{\omega_1^2}{2} - \frac{\omega_3^2}{2} = c_2'$$
$$9\omega_1^2 - \omega_3^2 = c_2.$$

at $t=0$, $\omega_1 = n$, $\omega_3 = 3n$.

$$9n^2 - 9n^2 = c_2$$
$$\Rightarrow c_2 = 0$$
$$\Rightarrow 9\omega_1^2 = \omega_3^2$$
$$3\omega_1 = \omega_3$$
$$\omega_3 = 3\sqrt{\frac{5n^2-\omega_2^2}{5}} \longrightarrow (10)$$

using these results, (8), (10) into (5)

$$3A\dot{\omega}_2 + 5A\omega_1\omega_3 = 0$$
$$3A\dot{\omega}_2 + 5A\left(\sqrt{\frac{5n^2-\omega_2^2}{5}}\right)\left(3\sqrt{\frac{5n^2-\omega_2^2}{5}}\right) = 0$$
$$3A\dot{\omega}_2 + 5A\left(3\left(\frac{5n^2-\omega_2^2}{5}\right)\right) = 0$$
$$3A\dot{\omega}_2 + 15A\left(\frac{5n^2-\omega_2^2}{5}\right) = 0$$
$$\frac{15A\dot{\omega}_2 + 15A(5n^2-\omega_2^2)}{5} = 0$$
$$15A\dot{\omega}_2 + 15A(5n^2-\omega_2^2) = 0$$
$$\dot{\omega}_2 + (5n^2-\omega_2^2) = 0$$

$$\dot{\omega}_2 - \omega_2^2 + 5n^2 = 0$$

$$\frac{d\omega_2}{dt} = \omega_2^2 - 5n^2.$$

$$d\omega_2 = (\omega_2^2 - 5n^2)dt$$

$$\frac{d\omega_2}{\omega_2^2 - 5n^2} = dt.$$

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$-\int \frac{d\omega_2}{5n^2 - \omega_2^2} = \int dt$$

$$-\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = t + c_1$$

$$= -\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = t + c_1$$

Put $t = 0, \omega_2 = 0$

$$c_1 = \frac{-1}{2\sqrt{5}n} \ln \frac{\cancel{\sqrt{5}n}}{\cancel{\sqrt{5}n}}$$

$$= \frac{-1}{2\sqrt{5}n} \ln(1) \to 0$$

$$c_1 = 0.$$

$$t = \frac{-1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2}$$

$$\frac{1}{2\sqrt{5}n} \ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = -t$$

$$\ln \frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = -2\sqrt{5}nt$$

$$\frac{\sqrt{5}n + \omega_2}{\sqrt{5}n - \omega_2} = e^{-2\sqrt{5}nt}$$

$$\frac{\cancel{\sqrt{5}n} + \cancel{\omega_2} + \sqrt{5}n - \cancel{\omega_2}}{\sqrt{5}\cancel{n} + \omega_2 - \cancel{\sqrt{5}n} + \omega_2}$$

$$= \frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1}$$

$$\frac{\cancel{2}\sqrt{5}n}{\cancel{2}\omega_2} = \frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1}$$

$$\omega_2 = \sqrt{5}n \left[\frac{e^{-2\sqrt{5}nt} - 1}{e^{-2\sqrt{5}nt} + 1} \right]$$

or also

$$\int \frac{d\omega_2}{\omega_2^2 - 5n^2} = \int dt$$

$$-\int \frac{d\omega_2}{5n^2 - \omega_2^2} = \int dt$$

$$-\int \frac{d\omega_2}{(\sqrt{5}n)^2 - (\omega_2)^2} = \int dt$$

$$\frac{-1}{\sqrt{5}n} \tanh^{-1} \frac{\omega_2}{\sqrt{5}n} = t$$

$$\tanh^{-1} \frac{\omega_2}{\sqrt{5}n} = -\sqrt{5}nt$$

$$\frac{\omega_2}{\sqrt{5}n} = -\tanh \sqrt{5}nt$$

$$\omega_2 = -\sqrt{5}n \tanh \sqrt{5}nt$$

$$\omega_2 = n\sqrt{5}\left[\frac{\cancel{e^{-\sqrt{5}nt}}}{\cancel{e^{-\sqrt{5}nt}}}\left[\frac{e^{-\sqrt{5}nt} - e^{\sqrt{5}nt}}{e^{-\sqrt{5}nt} + e^{\sqrt{5}nt}}\right]\right]$$

$$\omega_2 = n\sqrt{5}\tanh\sqrt{5}nt$$

Now $\omega_1 = \sqrt{\dfrac{5n^2 - (-\sqrt{5}n\tanh\sqrt{5}nt)^2}{5}}$

$$= \sqrt{\frac{5n^2 - 5n^2\tanh^2\sqrt{5}nt}{5}}$$

$$= n\sqrt{1 - \tanh^2\sqrt{5}nt}$$

$$= n\sqrt{\operatorname{sech}^2\sqrt{5}nt}$$

$$\omega_1 = n\operatorname{sech}\sqrt{5}nt$$

$$\Rightarrow \omega_3 = 3n\operatorname{sech}\sqrt{5}nt$$

$$\overline{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$$

$$= n\operatorname{sech}\sqrt{5}nt\,i - \sqrt{5}n\tanh\sqrt{5}nt\,j + 3n\operatorname{sech}\sqrt{5}nt\,k$$

$t \longrightarrow \infty$

$$\overline{\omega} = 0 - \sqrt{5}n(1) + 0 \qquad \left(\tanh\infty = 1,\ \operatorname{sech}\infty = 0\right)$$

$$= -n\sqrt{5}$$

$\overline{\omega} = \sqrt{5}n$ as Required.

Ans

formulas,

* $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\frac{a+x}{a-x}$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\frac{x-a}{x+a}$

or $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}$ (using