Hafiz Naveed Islam MMF18E049.

# Mathod Of Regularization.

The regularization method was established independently by Tikhonov and phillips. The regularization method consists of transforming first Kind integral equation to second Kind integral equation.

Before describing the Concept of regularization method, we define some familier definitions which may help us to understa-and the method of Regularization in best way.

## * Integral Equation.

An equation which contains an integral is called integral equation.

**example#**

$$g(x)\,y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)\,dt$$

## * Integrodifferential Equation

An equation which contains an integral and derivative of a function is called integrodifferential equation.

**example#**

$$u'(x) = f'(x) + \lambda\int_a^b K(x,t)\,u(t)\,dt$$

## * Linear integral equation

An equation of the form

$$g(x)\,y(x) = f(x) + \lambda\int_a K(x,t)\,y(t)\,dt$$

where $K(x,t)$ is the kernal of equation is called Linear integral equation.

## * Fredholm integral Equation

A linear integral equation of the following form

$$(*)\quad g(x)\,y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)\,dt$$

is known Fredholm integral equation of $3^{rd}$ - kind.
where a, b are constants and $f(x)$, $g(x)$ & $K(x,t)$ are known functions while $\lambda$ is a non-zero parameter and $K(x,t)$ is kernal.

## * Kinds of Fredholm integral equation

consider $g(x)\,y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt$ —— ①

**i)** If $g(x) = 0$, then (1) is said to be Fredholm integral equation of first kind. i.e

$$0 = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt$$

**ii)** If $g(x) = 1$ in ①, then ① is said to be Fredholm integral equation of $2^{nd}$ - kind i.e

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt$$

**iii)** If $g(x) \neq 0$ in ①, then ① is said to be Fredholm integral equation of $3^{rd}$ - kind.
i.e

$$g(x)\,y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,dt.$$

Now, we are able to describe the method of Regularization.

The regularization method transforms the linear Fredholm integral equation of first kind

$$f(x) = \int_a^b K(x,t)\, u(t)\, dt \quad \text{(A)}$$

to the approximation Fredholm integral equation

$$\alpha U_\alpha(x) = f(x) - \int_a^b K(x,t)\, u(t)\, dt \quad \text{(B)}$$

where $\alpha$ is a small positive parameter called regularization parameter. It is clear that equation (B) is an integral equation of the second kind that can be written as

$$U_\alpha(x) = \frac{1}{\alpha} f(x) - \frac{1}{\alpha} \int_a^b K(x,t)\, U_\alpha(t)\, dt \quad \text{(C)}$$

Moreover, it was proved by Tikhonov and phillips that the Solution $U_\alpha$ of equ(c) Converges to the solution $U(x)$ of equ (A) as $\alpha \to 0$. In other words, it was shown at

$$U(x) = \lim_{\alpha \to 0} U_\alpha(x)$$

**Important note.**

It is important to note that the Fredholm integral equation of the first Kind is ill-posed problem. The solution for an ill-posed problem may not exist, and if it exists it may not be unique.

As stated before, we will apply the regularization method to transform the first Kind Fredholm integral equation to the second kind integral equation. The resulting second kind integral equation will be then solved by well-known existing techniques such as successive approximation method and Direct computation method etc.

## Example # 01 :

Combine the method of Regularization and the direct computation method to solve the Fredholm integral Equation of first kind

$$\frac{1}{4} e^{x} = \int_0^{1/4} e^{x-t} u(t)\, dt \quad \text{—(A)}$$

## Solution :

Using the method of regularization, we transform Eq (A) as

$$u_\mu(x) = \frac{1}{4\mu} e^{x} - \frac{1}{\mu} \int_0^{1/4} e^{x-t} u_\mu(t)\, dt \quad \text{—(B)}$$

Now for further solution, we use direct computation method.

Then by direct computation method, equation (B) can be transformed as

$$u_\mu(x) = \frac{1}{4\mu} e^{x} - \frac{1}{\mu} \int_0^{1/4} e^{x} \cdot e^{-t} u_\mu(t)\, dt$$

$$u_\mu(x) = \left[\frac{1}{4\mu} - \frac{1}{\mu}\int_0^{1/4} e^{-t}\, u_\mu(t)\,dt\right] e^{x}$$

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right) e^{x} \quad \text{—— (C)}$$

where $\alpha = \int_0^{1/4} e^{-t}\, u_\mu(t)\,dt$ —— (D)

To compute $\alpha$, we use (C) in (D).
Then

$$\alpha = \int_0^{1/4} e^{-t}\left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right) e^{t}\,dt$$

$$\alpha = \int_0^{1/4} \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right) e^{t-t}\,dt$$

$$\alpha = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\int_0^{1/4} e^{0}\,dt$$

$$\alpha = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\left(t\Big|_0^{1/4}\right)$$

$$\alpha = \frac{1}{4}\left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)$$

$$\alpha = \frac{1}{16\mu} - \frac{\alpha}{4\mu}$$

$$\alpha + \frac{\alpha}{4\mu} = \frac{1}{16\mu}$$

$$\alpha\left(1 + \frac{1}{4\mu}\right) = \frac{1}{16\mu}$$

$$\alpha = \frac{1}{16\mu} \cdot \frac{4\mu}{4\mu+1}$$

$$\boxed{\alpha = \frac{1}{4(1+4\mu)}}$$

Using this in Eq 6, we get

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{1}{4\mu(1+4\mu)}\right)e^x$$

$$u_\mu(x) = \left(\frac{1+4\mu-1}{4\mu(1+4\mu)}\right)e^x$$

$$U_\mu(x) = \left(\frac{4\mu}{4\mu(1+4\mu)}\right) e^x$$

$$\boxed{U_\mu(x) = \frac{e^x}{1+4\mu}}$$

As $\mu \to 0$, we get exact solution $U(x)$.

So,

$$U(x) = \lim_{\mu \to 0} \frac{e^x}{1+4\mu}$$

$$U(x) = \frac{e^x}{1}$$

$$\boxed{U(x) = e^x}$$

Date:

S M T W T F S

## Example # 02 :

Combine method of Regularization and the Adomian decomposition method to solve Fredholm integral equation of First kind,

$$\frac{1}{3}e^{-x} = \int_0^{1/3} e^{t-x}\, u(t)\,dt \quad \text{——(A)}$$

## Solution :

Using Method of Regularization, we can transform eq (A) into Fredholm integral equation of second kind as

$$u_\mu(x) = \frac{1}{3\mu} e^{-x} - \frac{1}{\mu}\int_0^{1/3} e^{t-x} u_\mu(t)\,dt \quad \text{—(B)}$$

Now using Adomian decomposition method, we can write the unknow function $u_\mu(x)$ as sum of infinite number of components, so that eq (B) becomes, Date: ____________ S M T W T F S

$$\sum_{n=0}^{\infty} U_{\mu_n}(x) = \frac{1}{3\mu}e^{-x} - \frac{1}{\mu}\int_0^{1/3} e^{t-x}\sum_{n=0}^{\infty} U_{\mu_n}(t)\,dt$$

$$U_{\mu_0}(x) + U_{\mu_1}(x) + \cdots = \frac{1}{3\mu}e^{-x} - \frac{1}{\mu}\int_0^{1/3} e^{t-x}\left[U_{\mu_0}(t) + U_{\mu_1}(t) + \cdots\right]dt \quad -\text{(C)}$$

where zero$^{th}$ component equals the term not included under integral sign i.e

$$U_{\mu_0}(x) = \frac{1}{3\mu}e^{-x}$$

and we get the Recurrence relation as

$$U_{\mu_{k+1}}(x) = \frac{-1}{\mu}\int_0^{1/3} e^{t-x}\,U_{\mu_k}(t)\,dt \quad , k \geq 0$$

So, that

$$U_{\mu_1}(x) = \frac{-1}{\mu}\int_0^{1/3} e^{t-x}\,U_{\mu_0}(t)\,dt$$

$$U_{\mu_1}(x) = \frac{-1}{\mu}\int_0^{1/3} e^{t-x}\cdot\frac{1}{3\mu}e^{-t}\,dt$$

$$U_{\mu_1}(x) = \frac{-1}{\mu}\int_0^{1/3} \frac{e^{-x}}{3\mu}\,dt$$

Date: ______

$$U_{\mu_1}(x) = \frac{-1}{\mu} \cdot \frac{e^{-x}}{3\mu} \int_0^{1/3} dt$$

$$U_{\mu_1}(x) = \frac{-e^{-x}}{3\mu^2} \left| t \right|_0^{1/3}$$

$$\boxed{U_{\mu_1}(x) = \frac{-e^{-x}}{9\mu^2}}$$

And $U_{\mu_2}(x) = \frac{-1}{\mu} \int_0^{1/3} e^{t-x} \cdot \left( \frac{-e^{-\cancel{t}}}{9\mu^2} \right) dt$

$$U_{\mu_2}(x) = \frac{+e^{-x}}{9\mu^3} \int_0^{1/3} dt$$

$$U_{\mu_2}(x) = \frac{e^{-x}}{9\mu^3} \left| t \right|_0^{1/3}$$

$$\boxed{U_{\mu_2}(x) = \frac{e^{-x}}{27\mu^3}}$$

And $U_{\mu_3}(x) = \frac{-1}{\mu} \int_0^{1/3} e^{t-x} \left( \frac{e^{-x}}{27\mu^3} \right) dt$

$$U_{\mu_3}(x) = \frac{-e^{-x}}{27\mu^4} \int_0^{1/3} dt$$

$$U_{\mu_3}(x) = \frac{-e^{-x}}{27\mu^4}\,|t|_0^{1/3}$$

$$\boxed{U_{\mu_3}(x) = \frac{-e^{-x}}{81\mu^4}}$$

and so on.

Now, using these values of components of $\sum_{n=0}^{\infty} U_{\mu_n}(x)$, we have

$$U_\mu(x) = \sum_{n=0}^{\infty} U_{\mu_n}(x)$$

$$U_\mu(x) = \frac{-1}{3\mu}e^{-x} + \frac{1}{9\mu^2}e^{-x} - \frac{1}{27\mu^3}e^{-x} + \frac{1}{81\mu^4}e^{-x} - \cdots$$

$$U_\mu(x) = \frac{-1}{3\mu}e^{-x}\left[1 - \frac{1}{3\mu} + \frac{1}{9\mu^2} - \frac{1}{27\mu^3} + \cdots\right]$$

using $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

we have

$$U_\mu(x) = \frac{-1}{3\mu}e^{-x}\left(\frac{1}{1+1/3\mu}\right)$$

ate: ________ S M T W T F S

$$U_\mu(x) = \frac{-e^{-x}}{3\mu}\left(\frac{3\mu}{1+3\mu}\right)$$

$$\boxed{U_\mu(x) = \frac{-e^{-x}}{1+3\mu}}$$

Now as $\mu \to 0$, we get exact solution $U(x)$.

So,

$$U(x) = \lim_{\mu \to 0}\left(\frac{-e^{-x}}{1+3\mu}\right)$$

$$\boxed{U(x) = -e^{-x}}$$ Ans

Date:

S M T W T F S

## Example #03:

Combine method of Regularization and successive approximation method to solve Fredholm integral Equation of First kind,

$$\frac{1}{4}x = \int_0^1 xt\, u(t)\,dt \quad \text{—(A)}$$

## Solution:

Using Method of Regularization, we transform Eq (A) into Fredholm Integral Equation of second kind as

$$U_\mu(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, U_\mu(t)\,dt \quad \text{—(B)}$$

Now, to use successive approximation method, we let $U_{\mu_0}(x) = 0$ and introduce the recurrence relation as

$$U_{\mu_{k+1}}(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, U_{\mu_k}(t)\,dt \quad , k \geq 0$$

Date:  S M T W T F S

Then For $U_{\mu_1}(x)$, we have

$$U_{\mu_1}(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, U_{\mu_0}(t)\,dt$$

$$\boxed{U_{\mu_1}(x) = \frac{1}{4\mu}x} \qquad \because U_{\mu_0}(t) = 0$$

For $U_{\mu_2}(x)$, we have

$$U_{\mu_2}(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, U_{\mu_1}(t)\,dt$$

$$U_{\mu_2}(x) = \frac{1}{4\mu}x - \frac{1}{\mu}x\int_0^1 t\left(\frac{1}{4\mu}t\right)dt$$

$$U_{\mu_2}(x) = \frac{1}{4\mu}x - \frac{x}{\mu}\cdot\frac{1}{4\mu}\int_0^1 t^2\,dt$$

$$U_{\mu_2}(x) = \frac{x}{4\mu} - \frac{x}{4\mu^2}\left|\frac{t^3}{3}\right|_0^1$$

$$\boxed{U_{\mu_2}(x) = \frac{x}{4\mu} - \frac{x}{12\mu^2}} \Rightarrow \boxed{\frac{x}{4\mu}\left(1 - \frac{1}{3\mu}\right)}$$

Date: ______ S M T W T F S

For $U_{\mu_3}(x)$, we have

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{1}{\mu}\int_0^1 xt\, u_{\mu_2}(t)\,dt$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{1}{\mu}\int_0^1 xt\left(\frac{t}{4\mu} - \frac{t}{12\mu^2}\right)dt$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{x}{\mu}\int_0^1\left[\frac{t^2}{4\mu} - \frac{t^2}{12\mu^2}\right]dt$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{x}{4\mu^2}\int_0^1 t^2\,dt + \frac{x}{12\mu^3}\int_0^1 t^2\,dt$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{x}{4\mu^2}\left|\frac{t^3}{3}\right|_0^1 + \frac{x}{12\mu^3}\left|\frac{t^3}{3}\right|_0^1$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{x}{4\mu^2}\cdot\frac{1}{3} + \frac{x}{12\mu^3}\cdot\frac{1}{3}$$

$$U_{\mu_3}(x) = \frac{x}{4\mu} - \frac{x}{12\mu^2} + \frac{x}{36\mu^3}$$

Date:

S M T W T F S

and it can be written as

$$U_{\mu_3}(x) = \frac{x}{4\mu}\left[1 - \frac{1}{3\mu} + \frac{1}{9\mu^2}\right]$$

and generalising this so on, we get

$$U_{\mu_{k+1}}(x) = \frac{x}{4\mu}\left[1 - \frac{1}{3\mu} + \frac{1}{9\mu^2} - - - -\right]$$

$$U_{\mu_{k+1}}(x) = \frac{x}{4\mu}\left[\frac{1}{1 + 1/3\mu}\right]$$

$$U_{\mu}(x) = \frac{x}{4\mu} \cdot \frac{3\mu}{1+3\mu}$$

$$U_{\mu}(x) = \frac{3x}{4(1+3\mu)}$$

For exact solution, we let $\mu \to 0$

Then

$$U(x) = \lim_{\mu \to 0}\left[\frac{3x}{4(1+3\mu)}\right] = \boxed{\frac{3}{4}x}$$ Ans

Q.NO.1

$$\frac{1}{2}(1-e^{-2})e^{3x} = \int_0^1 e^{3x-4t}u(t)\,dt$$

Solution.

By using the method of regularization we have,

$$u_\mu(x) = \frac{1}{2\mu}(1-e^{-2})e^{3x} - \frac{1}{\mu}\int_0^1 e^{3x-4t}u(t)\,dt$$

Now by using direct computation method, we have

$$u_\mu(x) = \left(\frac{(1-e^{-2})}{2\mu} - \frac{\alpha}{\mu}\right)e^{3x} \quad \text{——} (1)$$

where

$$\alpha = \int_0^1 e^{-4t}u_\mu(t)\,dt \quad \text{——} (2)$$

Now using eq. (1) in (2).

$$\alpha = \int_0^1 e^{-4t}\left(\frac{(1-e^{-2})}{2\mu} - \frac{\alpha}{\mu}\right)e^{3t}dt$$

$$= \left(\frac{1-e^{-2}}{2\mu} - \frac{\alpha}{\mu}\right)\int_0^1 e^{-t}\,dt$$

$$= \left(\frac{1-e^{-2}}{2\mu} - \frac{\alpha}{\mu}\right)\left(-e^{-t}\right|_0^1$$

$$\alpha = \left(\frac{1-e^{-2}}{2\mu} - \frac{\alpha}{\mu}\right)(1-e^{-1})$$

$$\frac{\alpha}{(1-e^{-1})} = \frac{(1-e^{-2})}{2\mu} - \frac{\alpha}{\mu}$$

$$\frac{\alpha}{(1-e^{-1})} + \frac{\alpha}{\mu} = \frac{(1-e^{-2})}{2\mu}$$

$$\alpha\left(\frac{1}{1-e^{-1}} + \frac{1}{\mu}\right) = \frac{(1-e^{-2})}{2\mu}$$

$$\alpha\left(\frac{\mu+(1-e^{-1})}{(1-e^{-1})\mu}\right)=\frac{(1-e^{-2})}{2\mu}$$

$$\alpha\left(\frac{\mu+(1-e^{-1})}{(1-e^{-1})}\right)=\frac{(1-e^{-2})}{2}$$

$$\alpha=\frac{(1-e^{-2})(1-e^{-1})}{2(\mu+(1-e^{-1}))}$$

From eq. (1)

$$u_\mu(x)=\left(\frac{(1-e^{-2})}{2\mu}-\frac{(1-e^{-2})(1-e^{-1})}{2\mu(\mu+(1-e^{-1}))}\right)e^{3x}$$

$$=\left(\frac{(1-e^{-2})(\mu+(1-e^{-1}))-(1-e^{-2})(1-e^{-1})}{2\mu(\mu+(1-e^{-1}))}\right)e^{3x}$$

$$=\left(\frac{(1-e^{-2})\mu+\cancel{(1-e^{-2})(1-e^{-1})}-\cancel{(1-e^{-2})(1-e^{-1})}}{2\mu(\mu+(1-e^{-1}))}\right)e^{3x}$$

$$=\left[\frac{(1-e^{-2})\mu}{2\mu(\mu+(1-e^{-1}))}\right]e^{3x}$$

$$u_\mu(x)=\frac{(1-e^{-2})\,e^{3x}}{2(\mu+(1-e^{-1}))}$$

Now

$$u(x)=\lim_{\mu\to 0}u_\mu(x)=\lim_{\mu\to 0}\frac{(1-e^{-2})e^{3x}}{2(\mu+(1-e^{-1}))}$$

$$u(x)=\frac{(1-e^{-2})e^{3x}}{2(1-e^{-1})}$$

The required solution.

Q. NO. 2

$$\frac{1}{2}e^{3x} = \int_0^{1/2} e^{3x-3t}\, u(t)\, dt$$

Solution:

By using method of regularization

$$u_{\mu}(x) = \frac{1}{2\mu} e^{3x} - \frac{1}{\mu}\int_0^{1/2} e^{3x-3t} u_{\mu}(t)\, dt$$

Now we solve it by direct computation method.

$$u_{\mu}(x) = \left(\frac{1}{2\mu} - \frac{\alpha}{\mu}\right) e^{3x} \quad \text{—— (1)}$$

where

$$\alpha = \int_0^{1} e^{-3t} u_{\mu}(t)\, dt \quad \text{—— (2)}$$

Using eq. (1) in (2).

$$\alpha = \int_0^{1/2} e^{-3t}\left(\frac{1}{2\lambda} - \frac{\alpha}{\lambda}\right) e^{3t}\,dt$$

$$\alpha = \left(\frac{1}{2\lambda} - \frac{\alpha}{\lambda}\right)\int_0^{1/2} 1\,dt$$

$$\alpha = \left(\frac{1}{2\lambda} - \frac{\alpha}{\lambda}\right)\left(\frac{1}{2}\right)$$

$$2\alpha = \frac{1}{2\lambda} - \frac{\alpha}{\lambda}$$

$\Rightarrow$

$$2\alpha + \frac{\alpha}{\lambda} = \frac{1}{2\lambda} \Rightarrow \alpha\left(2 + \frac{1}{\lambda}\right) = \frac{1}{2\lambda}$$

$$\alpha\left(\frac{2\lambda+1}{\cancel{\lambda}}\right) = \frac{1}{2\cancel{\lambda}}$$

$$\alpha = \frac{1}{2(2\lambda+1)}$$

From (1)

$$u_\lambda(x) = \left(\frac{1}{2\lambda} - \frac{1}{2\lambda(2\lambda+1)}\right)e^{3x}$$

$$= \left(\frac{2\cancel{\lambda}+\cancel{1}-\cancel{1}}{2\cancel{\lambda}(2\lambda+1)}\right)e^{3x}$$

$$u_\lambda(x) = \frac{e^{3x}}{2\lambda+1}$$

Now

$$u(x) = \lim_{\lambda \to 0} u_\lambda(x) = \lim_{\lambda \to 0} \frac{e^{3x}}{2\lambda+1}$$

$$= \frac{e^{3x}}{0+1} = e^{3x}$$

$$\Rightarrow \quad u(x) = e^{3x} \qquad \text{Ans.}$$

## Q. NO. 3

$$\frac{3}{4}x = \int_0^1 x t^2 u(t)\,dt$$

**Solution.**

By using Regularization method

$$u_\mu(x) = \frac{3}{4\mu}x - \frac{1}{\mu}\int_0^1 x t^2 u_\mu(t)\,dt$$

Now by using direct computation method.

$$u_\mu(x) = \left(\frac{3}{4\mu} - \frac{\alpha}{\mu}\right)x \quad \text{--- (i)}$$

where $\alpha = \int_0^1 t^2 u_\mu(t)\,dt$ — (ii)

by using (i) in (ii) we have

$$\alpha = \int_0^1 t^2\left(\frac{3}{4\mu} - \frac{\alpha}{\mu}\right)t\,dt$$

$$= \left(\frac{3}{4\mu} - \frac{\alpha}{\mu}\right)\int_0^1 t^3\,dt = \left(\frac{3}{4\mu} - \frac{\alpha}{\mu}\right)\left|\frac{t^4}{4}\right|_0^1$$

$$\alpha = \left(\frac{3}{4\mu} - \frac{\alpha}{\mu}\right)\left(\frac{1}{4}\right)$$

$$\Rightarrow 4\alpha = \frac{3}{4\mu} - \frac{\alpha}{\mu}$$

$$\Rightarrow 4\alpha + \frac{\alpha}{\mu} = \frac{3}{4\mu}$$

$$\alpha\left(\frac{4\mu+1}{\mu}\right) = \frac{3}{4\mu}$$

$$\Rightarrow \alpha = \frac{3}{4(4\mu+1)}$$

From (i)

$$u_\mu(x) = \left(\frac{3}{4\mu} - \frac{3}{4\mu(4\mu+1)}\right)x$$

$$= \left(\frac{12\mu + \cancel{3} - \cancel{3}}{4\mu(4\mu+1)}\right)x$$

$$u_\mu(x) = \frac{3 \cdot x}{1+4\mu}$$

now

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \lim_{\mu \to 0} \frac{3x}{1+4\mu}$$

$$u(x) = 3x$$

## Q.NO.4

$$\frac{6}{5}x^2 = \int_0^1 x^2 t^2 u(t)\,dt$$

Solution

By using Regularization method,

$$u_\mu(x) = \frac{6}{5\mu}x^2 - \frac{1}{\mu}\int_0^1 x^2 t^2 u_\mu(t)\,dt$$

now by using Direct computation method, we have

$$u_\mu(x) = \left(\frac{6}{5\mu} - \frac{\alpha}{\mu}\right)x^2 \quad \text{——— (i)}$$

where

$$\alpha = \int_0^1 t^2 u_\mu(t)\,dt \quad \text{—— (ii)}.$$

now using (i) in (ii)

$$\alpha = \int_0^1 t^2 \left(\frac{6}{5\mu} - \frac{\alpha}{\mu}\right) t^2\,dt$$

$$= \left(\frac{6}{5\mu} - \frac{\alpha}{\mu}\right)\int_0^1 t^4\,dt$$

$$\alpha = \left(\frac{6}{5\mu} - \frac{\alpha}{\mu}\right)\left|\frac{t^5}{5}\right|_0^1 = \left(\frac{6}{5\mu} - \frac{\alpha}{\mu}\right)\left(\frac{1}{5}\right)$$

$$\Rightarrow \quad 5\alpha = \frac{6}{5\mu} - \frac{\alpha}{\mu}$$

$$\alpha\left(5 + \frac{1}{\mu}\right) = \frac{6}{5\mu}$$

$$\alpha\left(\frac{5\mu+1}{\mu}\right) = \frac{6}{5\mu}$$

$$\alpha = \frac{6}{5(5\mu+1)}$$

From ①

$$u_\mu(x) = \left(\frac{6}{5\mu} - \frac{6}{(5\mu+1)5\mu}\right)x^2$$

$$= \left(\frac{30\mu + \cancel{6} - \cancel{6}}{5\mu(5\mu+1)}\right)x$$

$$u_\mu(x) = \frac{6}{(5\mu+1)}x$$

Now

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \lim_{\mu\to 0} \frac{6x^2}{5\mu+1}$$

$$u(x) = 6x \quad \underline{Ans.}$$

Q.No 5

$$\frac{2}{5}x^2 = \int_{-1}^{1} x^2 t^2 u(t)\,dt$$

Solution.

By using Regularization method

$$u_\mu(x) = \frac{2}{5\mu}x^2 - \frac{1}{\mu}\int_{-1}^{1} x^2 t^2 u_\mu(t)\,dt$$

Now by Direct computation

$$u_\mu(x) = \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)x^2 \quad \text{——} \quad ①$$

where

$$\alpha = \int_{-1}^{1} t^2 u_\mu(t)\,dt \quad \text{——} \quad ②$$

Now using ① in ②

$$\alpha = \int_{-1}^{1} t^2\left(\frac{2}{5}\mu - \frac{\alpha}{\mu}\right)t^2\,dt$$

$$= \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)\int_{-1} t^4 dt$$

$$= \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)\Big|\frac{t^5}{5}\Big|_{-1}$$

$$= \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)\left(\frac{1}{5} + \frac{1}{5}\right)$$

$$\alpha = \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)\left(\frac{2}{5}\right)$$

$$\frac{5}{2}\alpha = \left(\frac{2}{5\mu} - \frac{\alpha}{\mu}\right)$$

$$\frac{5}{2}\alpha + \frac{\alpha}{\mu} = \frac{2}{5\mu}$$

$$\alpha\left(\frac{5}{2} + \frac{\cancel{\alpha}}{\mu}\right) = \frac{2}{5\mu}$$

$$\alpha\left(\frac{5\mu + 2}{2\cancel{\alpha}}\right) = \frac{2}{5\cancel{\mu}}$$

$$\alpha = \frac{4}{5(5\mu + 2)}$$

From ①

$$u_\mu(x) = \left(\frac{2}{5\mu} - \frac{4}{5\mu(5\mu+2)}\right)x^2$$

$$= \left(\frac{2(5\mu+2) - 4}{5\mu(5\mu+2)}\right)x^2$$

$$= \left(\frac{10\mu + \cancel{4} - \cancel{4}}{5\mu(5\mu+2}\right)x^2$$

$$u_\mu(x) = \frac{2x^2}{5\mu+2}$$

Now

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \lim_{\mu \to 0} \frac{2x^2}{5\mu+2}$$

$$= x^2 \qquad \text{Ans.}$$

Q.No.6

$$\frac{1}{5}x = \int_0^1 x\,t\,u(t)\,dt$$

**Solution:**

By using method of Regularization.

$$u_\mu(x) = \frac{1}{5\mu}x - \frac{1}{\mu}\int_0^1 x\,t\,u_\mu(t)\,dt$$

Now by Direct computation method, we have,

$$u_\mu(x) = \left(\frac{1}{5\mu} - \frac{\alpha}{\mu}\right)x \qquad (1)$$

wher

$$\alpha = \int_0^1 t\,u_\mu(t)\,dt \qquad (2)$$

Now using eq (1) in (2)

$$\alpha = \int_0^1 t\left(\frac{1}{5\mu} - \frac{\alpha}{\mu}\right)t\,dt$$

$$= \left(\frac{1}{5\mu} - \frac{\alpha}{\mu}\right)\int_0^1 t^2\,dt$$

$$= \left(\frac{1}{5\mu} - \frac{\alpha}{\mu}\right)\left|\frac{t^3}{3}\right|_0^1$$

$$\alpha = \left(\frac{1}{5\mu} - \frac{\alpha}{\mu}\right)(1/3)$$

$$3\alpha + \frac{\alpha}{\mu} = \frac{1}{5\mu}$$

$$\alpha\left(3 + \frac{1}{\mu}\right) = \frac{1}{5\mu}$$

$$\Rightarrow \alpha\left(\frac{3\mu + 1}{\mu}\right) = \frac{1}{5\mu}$$

$$\Rightarrow \alpha = \frac{1}{5(3\mu+1)}$$

Now from eq (1).

$$u_\mu(x) = \left(\frac{1}{5\mu} - \frac{1}{5\mu(3\mu+1)}\right)x$$

$$=\left(\frac{3u+1-1}{5u(3u+1)}\right)x$$

$$u_u(x) = \frac{3}{5(3u+1)}x$$

Now

$$u(x) = \lim_{u\to 0} u_u(x) = \lim_{u\to 0}\frac{3x}{5(3u+1)}$$

$$u(x) = \frac{3}{5}x \quad \text{Ans.}$$

## Q. No. 7

$$\frac{1}{6}x^2 = \int_0^1 x^2 t^2 u(t)\,dt$$

**Solution:**

Now by using method of Regularization,

$$u_u(x) = \frac{1}{6u}x^2 - \frac{1}{u}\int_0^1 x^2 t^2 u_u(t)\,dt$$

now by Direct computation method.

$$u_u(x) = \left(\frac{1}{6u} - \frac{\alpha}{u}\right)x^2 \quad \text{--- } (1)$$

where

$$\alpha = \int_0^1 t^2 u_u(t)\,dt \quad \text{--- } (2)$$

Now using (1) in (2).

$$\alpha = \int_0^1 t^2\left(\frac{1}{6u} - \frac{\alpha}{u}\right)t^2\,dt$$

$$= \left(\frac{1}{6u} - \frac{\alpha}{u}\right)\int_0^1 t^4\,dt$$

$$\alpha = \left(\frac{1}{6u} - \frac{\alpha}{u}\right)\left(\frac{1}{5}\right)$$

$$5\alpha + \frac{\alpha}{u} = \frac{1}{6u}$$

$$\alpha\left(\frac{5u+1}{u}\right) = \frac{1}{6u}$$

$$\alpha = \frac{1}{6(5\mu+1)}$$

Now from eq. (1).

$$u_\mu(x) = \left(\frac{1}{6\mu} - \frac{1}{6\mu(5\mu+1)}\right)x^2$$

$$= \left(\frac{5\mu+1-1}{6\mu(5\mu+1)}\right)x^2$$

$$u_\mu(x) = \frac{5}{6(5\mu+1)}x^2$$

Now

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \lim_{\mu\to 0}\frac{5}{6(5\mu+1)}x^2$$

$$= \frac{5}{6}x^2 \quad \text{Answer.}$$

**Q. No. 8**

$$\frac{2}{3}x^2 = \int_{-1}^{1} x^2 t^2 u(t)\,dt$$

**Solution:**

By using method of Regularization.

$$u_\mu(x) = \frac{2}{3\mu}x^2 - \frac{1}{\mu}\int_{-1}^{1} x^2 t^2 u_\mu(t)\,dt$$

Now by Direct computation method.

$$u_\mu(x) = \left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right)x^2 \quad \text{——— (1)}$$

where

$$\alpha = \int_{-1}^{1}\left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right)$$

$$\alpha = \int_{-1}^{1} t^2 u_\mu(t)\,dt \quad \text{——— (2)}$$

Using equation (1) in (2)

$$\alpha = \int_{-1}^{1} t^2\left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right)t^2\,dt$$

$$= \left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right)\int_{-1}^{1} t^4\,dt$$

$$\alpha = \left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right) \Big/ \left|\frac{1^5}{5}\right|$$

$$\alpha = \left(\frac{2}{3\mu} - \frac{\alpha}{\mu}\right)\left(\frac{2}{5}\right)$$

$$\frac{5}{2}\alpha = \frac{2}{3\mu} - \frac{\alpha}{\mu}$$

$$\frac{5}{2}\alpha + \frac{\alpha}{\mu} = \frac{2}{3\mu}$$

$$\alpha\left(\frac{5}{2} + \frac{1}{\mu}\right) = \frac{2}{3\mu}$$

$$\alpha\left(\frac{5\mu + 2}{2\mu}\right) = \frac{2}{3\mu}$$

$$\alpha = \frac{4}{3(5\mu+2)}$$

From (1)

$$u_\mu(x) = \left(\frac{2}{3\mu} - \frac{4}{3\mu(5\mu+2)}\right)x^2$$

$$= \left(\frac{10\mu + 4 - 4}{3\mu(5\mu+2)}\right)x^2$$

$$u_\mu(x) = \frac{10x^2}{3(5\mu+1)}$$

Now

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = \lim_{\mu \to 0} \frac{10x^2}{3(5\mu+1)}$$

$$u(x) = \frac{10}{3}x^2 \quad \text{Ans.}$$

## Q.No.9

$$-\frac{1}{4}x = \int_0^1 xt\,u(t)\,dt$$

**Solution:**

By using the method of Regularization

$$u_\mu(x) = -\frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 x\,t\,u_\mu(t)\,dt$$

By Direct computation method.

$$u_\mu(x) = \left(-\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)x \quad \text{——} \quad (1)$$

where

$$\alpha = \int_0^1 t\,u_\mu(t)\,dt \quad \text{——} \quad (2)$$

Now using eqn. (1) in (2)

$$\alpha = \int_0^1 t\left(-\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)t\,dt$$

$$= \left(-\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\int_0^1 t^2\,dt$$

$$\alpha = \left(-\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\left(\frac{1}{3}\right)$$

$$3\alpha + \frac{\alpha}{\mu} = -\frac{1}{4\mu}$$

$$\alpha\left(\frac{3\mu+1}{\mu}\right) = -\frac{1}{4\mu}$$

$$\alpha = \frac{-1}{4(3\mu+1)}$$

Now from eqn. (1)

$$u_\mu(x) = \left(-\frac{1}{4\mu} + \frac{1}{4\mu(3\mu+1)}\right)x$$

$$= \left(\frac{-3\mu - \cancel{1} + \cancel{1}}{4\mu(3\mu+1)}\right)x$$

$$u_\mu(x) = \frac{-3x}{4(3\mu+1)}$$

Now

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \lim_{\mu\to 0} \frac{-3x}{4(3\mu+1)}$$

$$u(x) = \frac{-3}{4}x \quad \text{Answer.}$$

Q.No.10

$$\frac{1}{4}x = \int_0^1 xt\, u(t)\, dt$$

Solution:

By using the method of Regularization. we have

$$u_\mu(x) = \frac{1}{4\mu}x - \frac{1}{\mu}\int_0^1 xt\, u_\mu(t)\, dt$$

Now by direct computation method we have

$$u_\mu(x) = \frac{1}{4\mu}x - \frac{\alpha}{\mu}x$$

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)x \quad \text{——} \quad (1)$$

where

$$\alpha = \int_0^1 t\,(u_\mu(t))\, dt \quad \text{—} \quad (2)$$

Using eq (1) in (2). we have

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)x$$

$$\alpha = \int_0^1 t\left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)t\, dt$$

$$= \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\int_0^1 t^2 dt$$

$$\alpha = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\left(\frac{t^3}{3}\right)_0^1$$

$$\alpha = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)\left(\frac{1}{3}\right)$$

$$3\alpha + \frac{\alpha}{\mu} = \frac{1}{4\mu}$$

13 $$\alpha\left(\frac{3\mu + 1}{\mu}\right) = \frac{1}{4\mu}$$

Thus $\alpha = \frac{1}{4(3\mu+1)}$

now by eq. ①

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{1}{4\mu(3\mu+1)}\right)x$$

$$= \left(\frac{3\mu+1-1}{4\mu(3\mu+1)}\right)x$$

$$u_\mu(x) = \frac{3}{4(3\mu+1)}x$$

Now

$$u(x) = \lim_{\mu\to 0} u_\mu(x) = \lim_{\mu\to 0} \frac{3x}{4(3\mu+1)}$$

$$u(x) = \frac{3}{4}x \quad \text{Ans.}$$

Date ________ Day M T W T F S

Q.11 $\frac{x}{12} = \int_0^1 x\,t\,u(t)\,dt$ —— (1)

Sol Using the method of regularization equation (1) can be transformed as

$$\alpha U_\alpha(x) = \frac{x}{12} - \int_0^1 x\,t\,u_\alpha(t)\,dt$$

or we can write as

$$u_\alpha(x) = \frac{x}{12\alpha} - \frac{1}{\alpha}\int_0^1 x\,t\,u_\alpha(t)\,dt \quad \text{—— (2)}$$

which is Fredholm Integral equation of second kind and can be solved by method of direct computation

So

$$u_\alpha(x) = \frac{x}{12\alpha} - \frac{x}{\alpha}\int_0^1 t\,u_\alpha(t)\,dt$$

$$u_\alpha(x) = \frac{x}{12\alpha} - \frac{x}{\alpha}\mu \quad \text{—— (3)}$$

where $\mu = \int_0^1 t\,u_\alpha(t)\,dt$ —— (4)

Now putting (3) in (4), we have

Date ______ ② Day M T W T F

$$\mu = \int_0^1 t\left(\frac{t}{12\alpha} - \frac{t}{\alpha}\mu\right) dt$$

$$\mu = \frac{1}{12\alpha}\int_0^1 t^2\,dt - \frac{\mu}{\alpha}\int_0^1 t^2\,dt$$

$$= \frac{1}{12\alpha}\left|\frac{t^3}{3}\right|_0^1 - \frac{\mu}{\alpha}\left|\frac{t^3}{3}\right|_0^1$$

$$= \frac{1}{36\alpha} - \frac{\mu}{3\alpha}$$

$$\mu + \frac{\mu}{3\alpha} = \frac{1}{36\alpha}$$

$$\mu\left(1 + \frac{1}{3\alpha}\right) = \frac{1}{36\alpha}$$

$$\mu\left(\frac{3\alpha+1}{3\alpha}\right) = \frac{1}{36\alpha}$$

$$\mu = \frac{1}{12(3\alpha+1)}$$

putting in eq 3

$$u_\alpha(x) = \frac{x}{12\alpha} - \frac{x}{\alpha}\left(\frac{1}{12(3\alpha+1)}\right)$$

$$= x\left[\frac{3\alpha+1-1}{12\alpha(3\alpha+1)}\right]$$

$$= x\left(\frac{3\alpha}{12\alpha(3\alpha+1)}\right)$$

Date ____________ Day 3

$$U_\alpha(x) = \frac{x}{4(3\alpha+1)}$$

Now

$$\lim_{\alpha\to 0} U_\alpha(x) = \lim_{\alpha\to 0} \frac{x}{4(3\alpha+1)}$$

So the exact solution of equation (1) is

$$u(x) = \lim_{\alpha\to 0} U_\alpha(x) = \frac{x}{4}$$

$$\Rightarrow \quad u(x) = \frac{x}{4}$$

Q.12 $$\frac{7x}{12} = \int_0^1 xt\, u(t)\, dt \quad \text{--- (1)}$$

Using the method of regularization eq (1) can be transformed as

$$\alpha U_\alpha(x) = \frac{7}{12x} - \int_0^1 xt\, U_\alpha(t)\, dt$$

or

Date ______ (4) Day M T W T F S

$$u_\alpha(x) = \frac{7x}{12\alpha} - \frac{1}{\alpha}\int_0^1 xt\, u_\alpha(t)\,dt$$

which is Fredholm Integral equation of 2nd kind. and can be solved by direct computation. So

$$u_\alpha(x) = \frac{7x}{12\alpha} - \frac{x}{\alpha}\int_0^1 t\, u_\alpha(t)\,dt$$

or

$$u_\alpha(x) = \frac{7x}{12\alpha} - \frac{x}{\alpha}M \quad \text{——— } (2)$$

where $M = \int_0^1 t\, u_\alpha(t)\, dt$ ——— (3)

putting (2) in (3)

$$M = \int_0^1 t\left(\frac{7t}{12\alpha} - \frac{t}{\alpha}M\right) dt$$

$$= \frac{7}{12\alpha}\int_0^1 t^2 dt - \frac{M}{\alpha}\int_0^1 t^2\, dt$$

$$= \frac{7}{12\alpha}\left|\frac{t^3}{3}\right|_0^1 - \frac{M}{\alpha}\left|\frac{t^3}{3}\right|$$

$$= \frac{7}{36\alpha} - \frac{M}{3\alpha}$$

Date ______ Day M T W T F S

$$\mu + \frac{\mu}{3\alpha} = \frac{7}{36\alpha}$$

$$\mu\left(1 + \frac{1}{3\alpha}\right) = \left(\frac{7}{36\alpha}\right)$$

$$\mu\left(\frac{3\alpha+1}{3\alpha}\right) = \frac{7}{36\alpha}$$

$$\mu = \frac{7}{12(3\alpha+1)}$$

putting in equation (2)

$$u_\alpha(x) = \frac{7x}{12\alpha} - \frac{x}{\alpha}\left(\frac{7}{12(3\alpha+1)}\right)$$

$$= 7x\left(\frac{3\alpha+1-1}{12\alpha(3\alpha+1)}\right)$$

$$= 7x\left(\frac{3\alpha}{12\alpha(3\alpha+1)}\right)$$

$$u_\alpha(x) = \frac{7x}{4(3\alpha+1)}$$

Now

$$\lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0} \frac{7x}{4(3\alpha+1)}$$

So the exact solution is

$$u(x) = \frac{7x}{4}$$

Date ______ (6) Day M T W T F S

Q.13 $$\frac{\pi}{2}\sin x = \int_0^{\pi} \cos(x-t)\, u(t)\, dt \quad \text{——} (1)$$

Using method of regularization we can transform (1) as

$$\mu\, u_\mu(x) = \frac{\pi}{2}\sin x - \int_0^{\pi} \cos(x-t)\, u_\mu(t)\, dt$$

or

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{1}{\mu}\int_0^{\pi} \cos(x-t)\, u_\mu(t)\, dt$$

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{1}{\mu}\int_0^{\pi} (\cos x\cos t + \sin x\sin t)\, u_\mu(t)\, dt$$

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{1}{\mu}\int_0^{\pi} \cos x\cos t\, u_\mu(t)\, dt - \frac{1}{\mu}\int_0^{\pi} \sin x\sin t\, u_\mu(t)\, dt$$

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{\cos x}{\mu}\int_0^{\pi} \cos t\, u_\mu(t)\, dt - \frac{\sin x}{\mu}\int_0^{\pi} \sin t\, u_\mu(t)\, dt$$

Date ______ (7) Day M T W T F S

$$u_\mu(x) = \frac{\pi}{2\mu}\sin x - \frac{\sin x}{\mu}\alpha - \frac{\cos x}{\mu}\beta \quad \text{--- (2)}$$

where

$$\alpha = \int_0^{\pi} \sin t\, u_\mu(t)\, dt \quad \text{--- (A)}$$

and $$\beta = \int_0^{\pi} \cos t\, u_\mu(t)\, dt \quad \text{--- (B)}$$

Equation (2) implies

$$u_\mu(x) = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin x - \frac{\beta}{\mu}\cos x \quad \text{--- (3)}$$

Now putting eq (3) in eq (A)

$$\alpha = \int_0^{\pi} \sin t \left[\left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin t - \frac{\beta}{\mu}\cos t\right] dt$$

$$\alpha = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\int_0^{\pi} \sin^2 t\, dt - \frac{\beta}{\mu}\int_0^{\pi} \sin t \cos t\, dt$$

$$= \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\int_0^{\pi} \frac{1-\cos 2t}{2}\, dt - \frac{\beta}{\mu}\left|\frac{\sin^2 t}{2}\right|_0^{\pi}$$

$$= \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\left(\frac{\pi}{2}\right) - 0$$

$$\alpha = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\frac{\pi}{2}$$

$$\alpha = \frac{\pi^2}{4\mu} - \frac{\alpha\pi}{2\mu}$$

$$\alpha + \frac{\alpha\pi}{2\mu} = \frac{\pi^2}{4\mu}$$

$$\alpha\left(1 + \frac{\pi}{2\mu}\right) = \frac{\pi^2}{4\mu}$$

$$\alpha\left(\frac{2\mu + \pi}{2\mu}\right) = \frac{\pi^2}{4\mu}$$

$$\alpha = \frac{\pi^2}{2(\pi + 2\mu)}$$

Now putting eq 3 in B

$$\beta = \int_0^{\pi} \cos t\left[\left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin t - \frac{\beta}{\mu}\cos t\right] dt$$

$$\beta = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\int_0^{\pi} \sin t \cos t \, dt - \frac{\beta}{\mu}\int_0^{\pi} \cos^2 t \, dt$$

$$\beta = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\left|\frac{\sin^2 t}{2}\right|_0^{\pi} - \frac{\beta}{\mu}\int_0^{\pi}\frac{1+\cos 2t\,dt}{2}$$

$$\beta = 0 - \frac{\beta}{\mu}\int_0^{\pi}\frac{1}{2}dt - \frac{\beta}{\mu}\int_0^{\pi}\frac{\cos 2t\,dt}{2}$$

$$\beta = -\frac{\beta}{\mu}\left(\frac{\pi}{2}\right) - \frac{\beta}{4\mu}\left|\sin 2t\right|_0^{\pi}$$

$$\beta = -\frac{\beta\pi}{2\mu} - \frac{\beta}{4\mu}(0)$$

$$\beta + \frac{\beta\pi}{2\mu} = 0$$

$$\Rightarrow \quad \beta = 0$$

Now putting value of $\alpha$ and $\beta$ in equation (3)

So

$$(3) \Rightarrow \quad u_\mu(x) = \left(\frac{\pi}{2\mu} - \frac{\alpha}{\mu}\right)\sin x - \frac{\beta}{\mu}\cos x$$

$$u_\mu(x) = \left(\frac{\pi}{2\mu} - \frac{\pi^2}{2\mu(\pi+2\mu)}\right)\sin x - 0$$

$$u_\mu(x) = \left(\frac{\pi(\pi+2\mu) - \pi^2}{2\mu(\pi+2\mu)}\right)\sin x$$

$$u_\mu(x) = \frac{\pi^2 + 2\pi\mu - \pi^2}{2\mu(\pi+2\mu)}\sin x$$

$$u_\mu(x) = \frac{\cancel{2}\pi\cancel{\mu}\sin x}{\cancel{2}\cancel{\mu}(\pi+2\mu)}$$

$$u_\mu(x) = \frac{\pi \sin x}{\pi+2\mu}$$

Now

$$\lim_{\mu \to 0} u_\mu(x) = \lim_{\mu \to 0} \frac{\pi \sin x}{\pi+2\mu}$$

$$= \frac{\cancel{\pi}\sin x}{\cancel{\pi}}$$

$$= \sin x$$

So the exact solution is

$$u(x) = \sin x$$